REPORT 837

METHODS FOR CALCULATING PULSED RADAR EMISSION SPECTRUM BANDWIDTH

(Study Programme 60A/1)

(1982)

1. Introduction

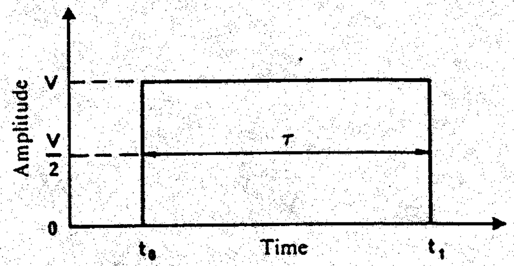
Recommendation 337 describes the primary factors which determine frequency separation between channels. These factors take into consideration the receiver selectivity and the transmitter emission spectrum. Report 654 describes a method of calculating adjacent band interference by taking into consideration the selectivity and the transmitter emission spectrum. In order to address technical issues relating to radar spectrum utilization, it would be useful to have available a means of calculating the radar emission spectrum based on radar technical characteristics. A method of easily relating spectral bounds to radar technical parameters is required. Various techniques have been developed to perform this type of analysis. The methods presented in this Report develop, through straightforward calculations, a straight line first-order approximation of the spectral boundaries of a trapezoidal radar pulse and a linear frequency modulated (FM), "chirped", radar pulse. The techniques outlined below are based on five major pulse characteristics which affect the radio spectrum emission characteristics of pulsed radars. These are the pulse amplitude, pulse duration, pulse rise time and fall time, and frequency modulation of the RF carrier during the pulse (chirping).

2. Calculation of radar emission spectrum

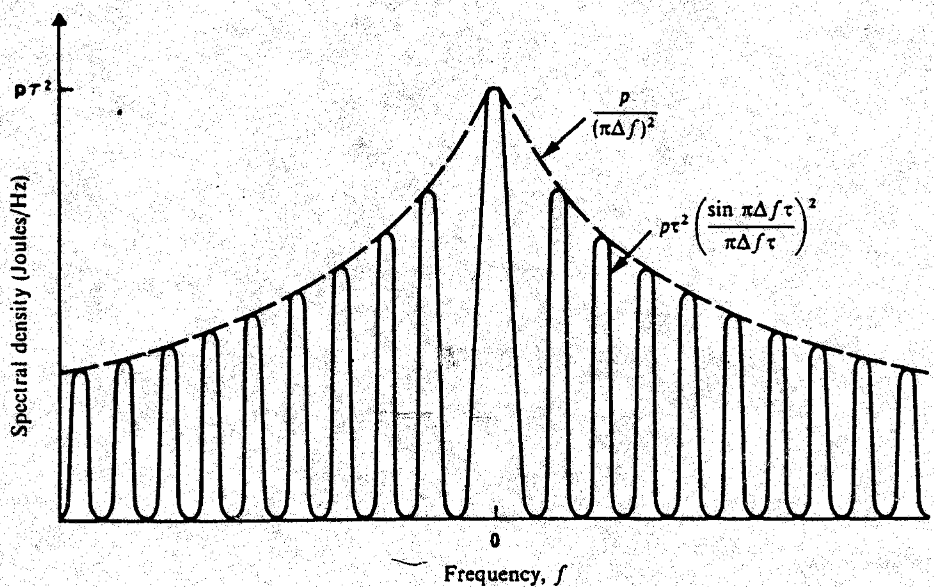
To precisely calculate the emission spectrum of a radar, the time domain waveform is transformed into the frequency domain through a Fourier transform. The theoretical radar pulse is an ideal rectangular pulse whose time and frequency domain representations are shown in Figs. 1a and 1b.

As a practical matter, a rectangular pulse cannot be obtained. The leading and trailing edges of the pulse will have a finite slope because of practical circuit design considerations. This results in a trapezoidal pulse shape (Fig. 2a). The Fourier transform of the square of the amplitude of the waveform yields the energy-density spectrum of the pulse. The spectrum takes the form shown by the solid curve in Fig. 2b.

In EMC analysis, it is appropriate as well as convenient to represent the spectrum by a bound, such as the dotted curve in Fig. 2b. It is common practice to normalize the graph of the spectral density by dividing by the peak value $p\tau^2$ (where p: peak pulse power level (W), and τ: pulse duration between half amplitude points (s)) and express the ratio in decibels (dB). This normalizing process yields the relative energy density in dB.

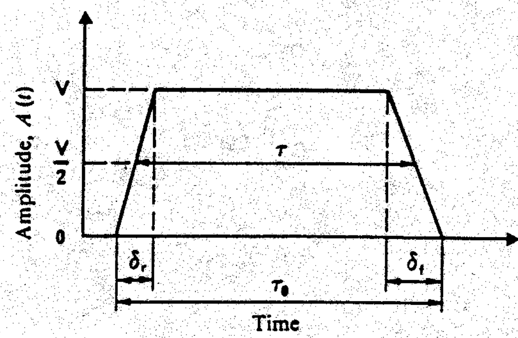


(a) Time domain

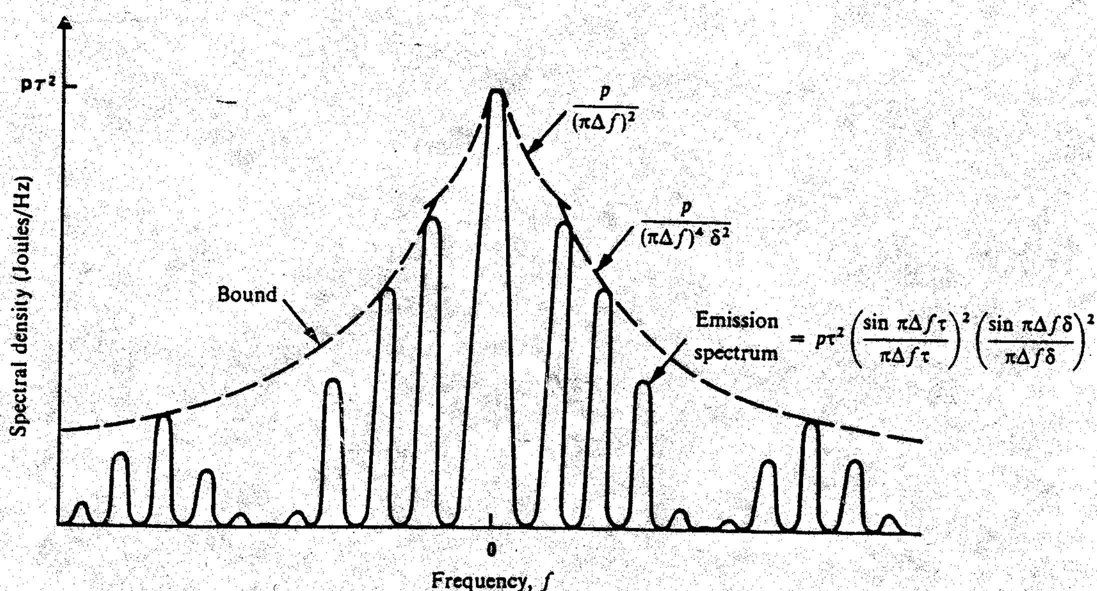


(b) Frequency domain

FIGURE 1 — *Rectangular pulse*



(a) Time domain



(b) Frequency domain

FIGURE 2 — *Trapezoidal pulse (without FM)*

The bounds on the spectrum of a trapezoidal pulse can be obtained by a simple calculation technique [Mason and Zimmerman, 1960]. This procedure involves four steps to draw an approximation of the bounds of the trapezoidal pulse spectrum (see Fig. 3). When a logarithmic scale is used for the abscissa (frequency) and a decibel, (dB) scale is used for the ordinate (relative to the peak spectral power density), the curves representing the bounds of the spectrum can be drawn as straight lines, as shown in Fig. 3. The frequency is referenced to f_c, i.e. $\Delta f = f - f_c$. To plot the spectrum bounds, the following steps are required:

Step 1: Calculate the critical frequencies, Δf_2 and Δf_3 using the equations below:

$$\Delta f_2 = \pm \frac{1}{\pi \tau} \tag{1a}$$

$$\Delta f_3 = \pm \frac{1}{\pi \sqrt{\tau \delta}} \qquad \text{where} \qquad \frac{1}{\delta} = \frac{1}{2}\left[\frac{1}{\delta_r} + \frac{1}{\delta_f}\right] \tag{1b}$$

where:

τ: pulse duration between half amplitude points, (s)

δ_r: pulse risetime, (s) } defined as the time interval from

δ_f: pulse falltime, (s) } 0 to 100% of the voltage-amplitude.

Note. – The spectrum is symmetrical about the carrier frequency (f_c) whether or not the pulse is symmetrical, (i.e. δ_r equal or not equal to δ_f).

Step 2: On semi-logarithmic paper draw line 1 horizontally through 0 dB.

Step 3: Starting on line 1 at Δf_2, draw line 2 with a slope of −20 dB/decade.

Step 4: Starting on line 1 at Δf_3, draw line 3 with a slope of −40 dB/decade.

The spectrum is bounded by lines 1, 2 and 3 of Fig. 3.

The peak energy density level (p_d), corresponding to the 0 dB level in Fig. 3, is calculated as follows:

$$p_d = (p\tau^2) \qquad \text{Joules/Hz} \tag{2}$$

where:

p: peak pulse power level (W).

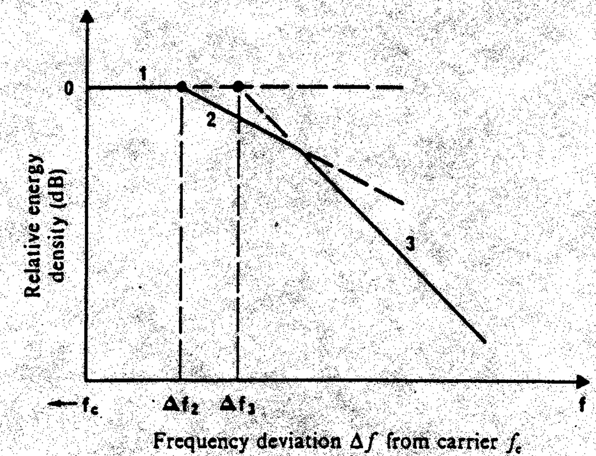


FIGURE 3 — *Spectrum bounds (Mason-Zimmerman approximation) of a trapezoidal pulse spectrum envelope*

When linear frequency modulation is applied to the pulsed carrier frequency as shown in Fig. 4, the peak of the spectrum is reduced in amplitude from $p\tau^2$ to $p\tau/\beta$ and spectrally broadened, but spectrum bounds further removed from the carrier frequency remain unaffected by the frequency modulation (Fig. 5). If the pulse is symmetrical about some time, t_0, (i.e. $\delta_r = \delta_f$) the spectrum will be symmetrical with reference to the nominal carrier f_c. If the pulse is asymmetrical, (i.e. $\delta_r \neq \delta_f$), the spectrum will be asymmetrical as shown in Fig. 6. The direction in which the spectrum is shifted will depend on whether the deviation is negative or positive with respect to time. The spectrum boundaries further removed from the carrier frequency will be symmetrical about a frequency, f_0, which is displaced from f_c, as described in equation (3) [Newhouse, 1982].

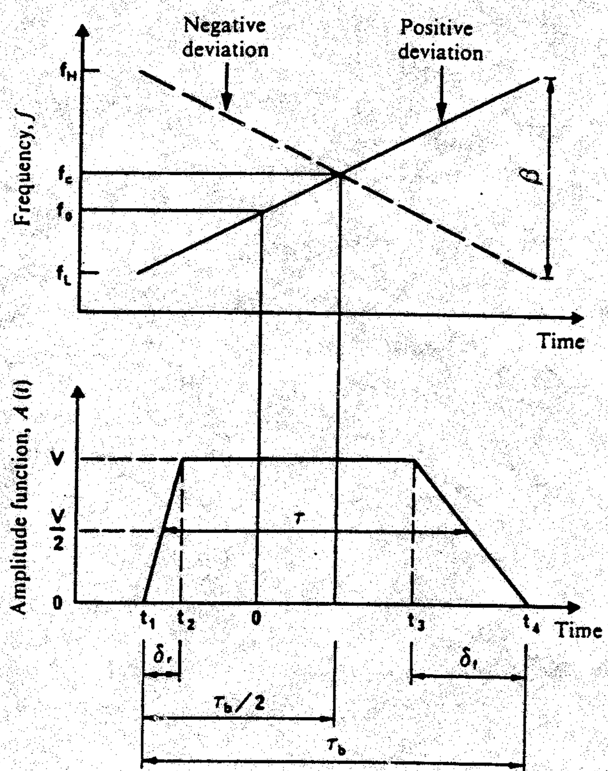


FIGURE 4 — *Parameters describing a chirp pulse*

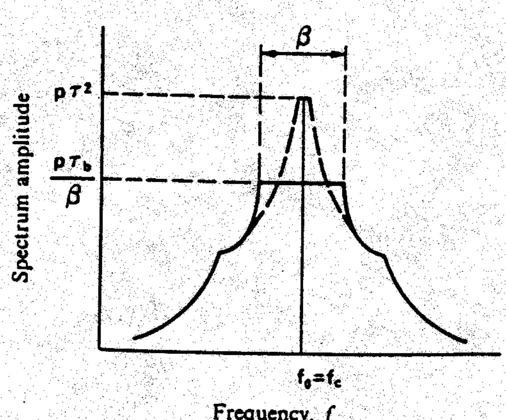


FIGURE 5 — *Shape of spectrum of chirp pulse when $\delta_r = \delta_f$*

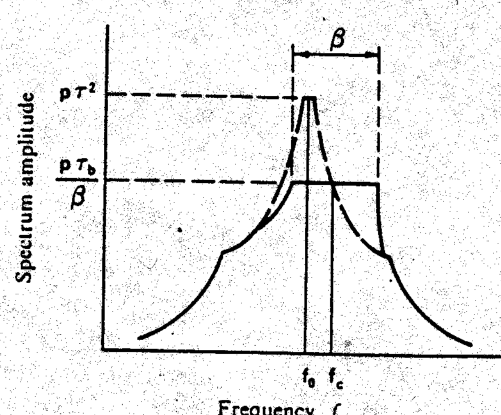


FIGURE 6 — *Shape of spectrum of chirp pulse when $\delta_r < \delta_f$ and frequency deviation is positive*

The energy-density spectrum is generally normalized by dividing by $p\tau/\beta$ and expressing the ratio in decibels to obtain the relative energy-density spectrum in dB,

$$f_0 = f_c + Q\left[\frac{\beta(\delta_r - \delta_f)}{2(\delta_r + \delta_f)}\right] \tag{3}$$

where Q is defined in Table I and β is defined in Fig. 4.

The mathematical description of the frequency spectrum of a FM pulse entails the use of Fresnel integrals. Evaluation of the integral is very tedious and must be performed with a high degree of precision to obtain meaningful results. Even when a computer is used the required precision may not be attained unless special care is taken in preparing the algorithm [Newhouse, 1973]. To overcome this difficulty, an approximate bound on the emission spectrum of a frequency-modulated pulse is developed here which is similar in technique to the procedure presented above. For the FM pulse, there are two cases which must be considered. First, when the product of $\beta\tau \leqslant 2/\pi$ the spectrum approximation reduces to the procedure presented above for the trapezoidal pulse. Second, when the product $\beta\tau > 2/\pi$, the procedure expands to seven steps and involves more complex calculations. The procedure for calculating the spectrum bounds for $\beta\tau > 2/\pi$ is presented below [Newhouse, 1982] and shown in Fig. 7. Step 1(A) is used when the pulse risetime is equal to the falltime ($\delta_r = \delta_f$). Step 1(B) is used when the pulse risetime is not equal to the falltime ($\delta_r \neq \delta_f$).

Step 1(A): If the pulse is symmetrical, ($\delta_r = \delta_f$) and $\beta\tau > 2/\pi$, calculate the critical frequencies, Δf_2, Δf_3, Δf_a, Δf_b as shown by equations (4a), (4b), (4c) and (4d):

$$\Delta f_2 = \pm\frac{1}{\pi}\left[\frac{\beta}{\tau_b}\right]^{1/2} \tag{4a}$$

$$\Delta f_3 = \pm\frac{1}{\pi}\left[\frac{\beta}{\tau_b}\right]^{1/4}\left[\frac{1}{\delta}\right]^{1/2} \quad \text{where} \quad \frac{1}{\delta} = \frac{1}{2}\left[\frac{1}{\delta_r} + \frac{1}{\delta_f}\right] \tag{4b}$$

$$\Delta f_{a\pm} = \pm\frac{\beta}{2}\left[1 - \frac{\delta}{\tau_b}\right] \tag{4c}$$

$$\Delta f_{b\pm} = 2\Delta f_{a\pm} \tag{4d}$$

where:

$\beta \quad = f_H - f_L$ from Fig. 4; and

$f_0 \quad = f_c = (f_H + f_L)/2$

The spectrum is now symmetrical about f_c, so the same plot can be used for positive and negative halves, i.e. $f_0 = f_c$.

Step 1(B): When the pulse is asymmetrical, ($\delta_r \neq \delta_f$) and $\beta\tau > 2/\pi$, critical frequencies may be calculated using the following equations:

$$\Delta f_2 = \pm\frac{1}{\pi}\left[\frac{\beta}{\tau_b}\right]^{1/2} \tag{5a}$$

$$\Delta f_3 = \pm\frac{1}{\pi}\left[\frac{\beta}{\tau_b}\right]^{1/4}\left[\frac{1}{\delta}\right]^{1/2} \quad \text{where} \quad \frac{1}{\delta} = \frac{1}{2}\left[\frac{1}{\delta_r} + \frac{1}{\delta_f}\right] \tag{5b}$$

When the spectrum is asymmetrical, positive and negative halves must be plotted separately, because $f_0 \neq f_c$ as shown in equation (3).

For positive Δf:

$$\Delta f_{a+} = \frac{M\beta}{\delta_r + \delta_f}\left[1 - \frac{\delta_r + \delta_f}{2\tau_b}\right] \tag{6a}$$

$$\Delta f_{b+} = \begin{cases} 2\Delta f_{a\pm} & (\text{when } \beta\delta \leqslant 1/\pi) \\ \dfrac{M\beta}{\delta_r + \delta_f}\left[\dfrac{1}{1 - \sqrt{\dfrac{N}{2(\delta_r + \delta_f)}}}\right] & (\text{when } \beta\delta > 1/\pi) \end{cases} \tag{6b}$$

For negative Δf:

$$\Delta f_{a-} = \frac{-N\beta}{\delta_r + \delta_f}\left[1 - \frac{\delta_r + \delta_f}{2\tau_b}\right] \tag{7a}$$

$$\Delta f_{b-} = \begin{cases} 2\Delta f_{a-} & \text{(when } \beta\delta \leqslant 1/\pi) \\ \dfrac{-N\beta}{\delta_r + \delta_f}\left[\dfrac{1}{1 - \sqrt{\dfrac{M}{2(\delta_r + \delta_f)}}}\right] & \text{(when } \beta\delta > 1/\pi) \end{cases} \tag{7b}$$

For M, N and Q values used here, see Table I.

Step 2: Draw line 1 horizontally through 0 dB (Figs. 7 and 8).

Step 3: If Δf_b is less than $1/(\pi\delta)$, use Fig. 7 and draw line 2 with a slope of -20 dB/decade starting on line 1 at Δf_2.

If Δf_b is equal to or greater than $1/(\pi\delta)$, skip this step and use Fig. 8 for the next step.

Step 4: Draw line 3 with a slope of -40 dB/decade starting on line 1 at Δf_3 in Fig. 7 or Fig. 8.

Step 5: Locate point "a" or "a'" at 6 dB down from 0 dB at Δf_a or $\Delta f_a'$ in Fig. 7 or Fig. 8 respectively.

Step 6: Locate point "b" or "b'" at Δf_b or $\Delta f_b'$ on line 2 in Fig. 7 or on line 3 in Fig. 8 respectively.

Step 7: Draw line 4 or line 4' through points "a" and "b" in Fig. 7 or through points "a'" and "b'" in Fig. 8.

TABLE I — *M, N and Q values for equations (3), (6) and (7)*

Constants	Positive frequency shift	Negative frequency shift
M	δ_f	δ_r
N	δ_r	δ_f
Q	+1	−1

Spectrum bounds: For the situation where $\Delta f_b < 1/(\pi\delta)$, the spectrum is bounded by a curve as described by lines 1, 4, 2 and 3 in Fig. 7. If $\Delta f_b \geqslant 1/(\pi\delta)$, the spectrum would be bounded by a curve as described by lines 1, 4 and 3 in Fig. 8.

The peak energy density level (p_d) corresponding to the 0 dB level in Figs. 7 or 8 is calculated as follows:

$$p_d = (p\tau_b/\beta) \qquad \text{Joules/Hz} \tag{8}$$

where:

β : frequency deviation during pulse (Hz),

τ_b : pulse width at the base of the pulse (s).

The techniques presented above are methods to calculate a first order approximation of a pulsed radar emission spectrum. The bounds obtained with these techniques are within one dB of the peaks of the main lobe and major side lobes of the pulsed emission spectrum envelope. These techniques are a reasonable approximation of the emission spectrum boundary up to a value of Δf corresponding to a spectral power density between 60 and 70 dB below the peak energy level. In this region, noise and distortion generated in the high power output stages create a noise floor which dominates the spectrum bounds for larger values of Δf.

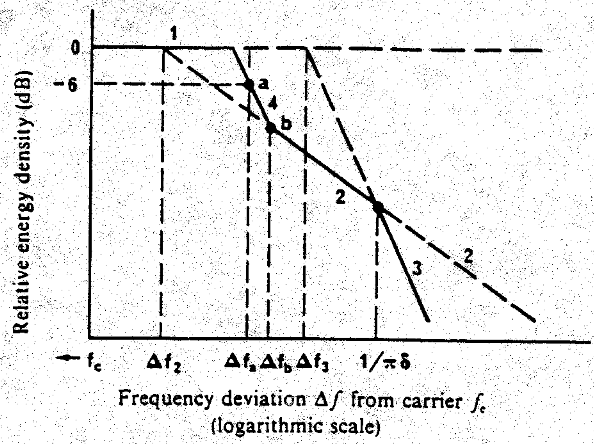


FIGURE 7 — *Spectrum bounds for a linear FM pulse when* $\Delta f_b < \frac{1}{\pi\delta}$

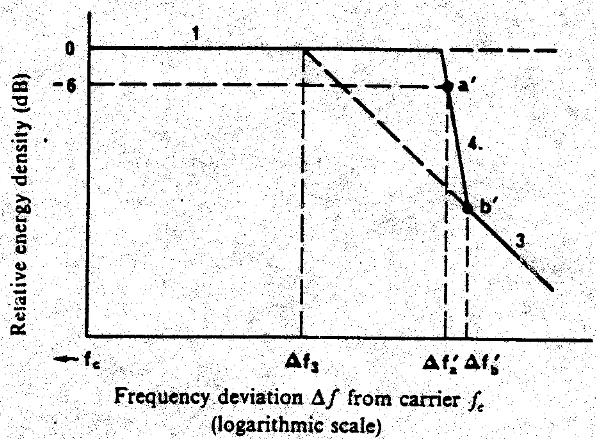


FIGURE 8 — *Spectrum bounds for a linear FM pulse when* $\Delta f_b' \geq \frac{1}{\pi\delta}$

3. Sample calculations of radar emission spectrum envelope

Example 1. – Trapezoidal pulse

$\tau = 6 \times 10^{-6}$ s
$\delta_r = 0.2 \times 10^{-6}$ s
$\delta_f = 0.35 \times 10^{-6}$ s

Step 1: Calculate critical frequencies Δf_2 and Δf_3 using equations (1a) and (1b).

$$\Delta f_2 = \pm \frac{1}{\pi\tau} = \pm \frac{1}{\pi\,(6 \times 10^{-6})} = 53.1 \text{ kHz}$$

$$\frac{1}{\delta} = \frac{1}{2}\left[\frac{1}{\delta_r} + \frac{1}{\delta_f}\right] = \frac{10^6}{2}\left[\frac{1}{0.2} + \frac{1}{0.35}\right] = \frac{1}{0.25 \times 10^{-6}}$$

$$\Delta f_3 = \pm \frac{1}{\pi\,(\tau\delta)^{1/2}} = \frac{1}{\pi\,[(6 \times 10^{-6})\,(0.25 \times 10^{-6})]^{1/2}} = 259.9 \text{ kHz}$$

Step 2: In Fig. 9, draw line 1 horizontally through 0 dB on the ordinate representing the peak energy level.

Step 3: Starting on line 1 at Δf_2 draw line 2 with a slope of -20 dB/decade.

Step 4: Starting on line 1 at Δf_3 draw line 3 with a slope of -40 dB/decade.

The spectrum bounds are described by lines 1, 2 and 3. To determine the accuracy of this approximation technique, the emission spectrum envelope was calculated. The calculated spectrum is plotted on Fig. 9 using the symbol "·" to designate representative points on the spectral power density function.

The peak energy density level corresponding to the 0 dB point is calculated as follows:

Assume: peak power (p) = 1×10^6 W, and

from equation (2):

$$\begin{aligned} p_d &= (p\tau^2) \\ &= (1 \times 10^6)(6 \times 10^{-6})^2 \\ &= 3.6 \times 10^{-5} \quad \text{Joules/Hz.} \end{aligned}$$

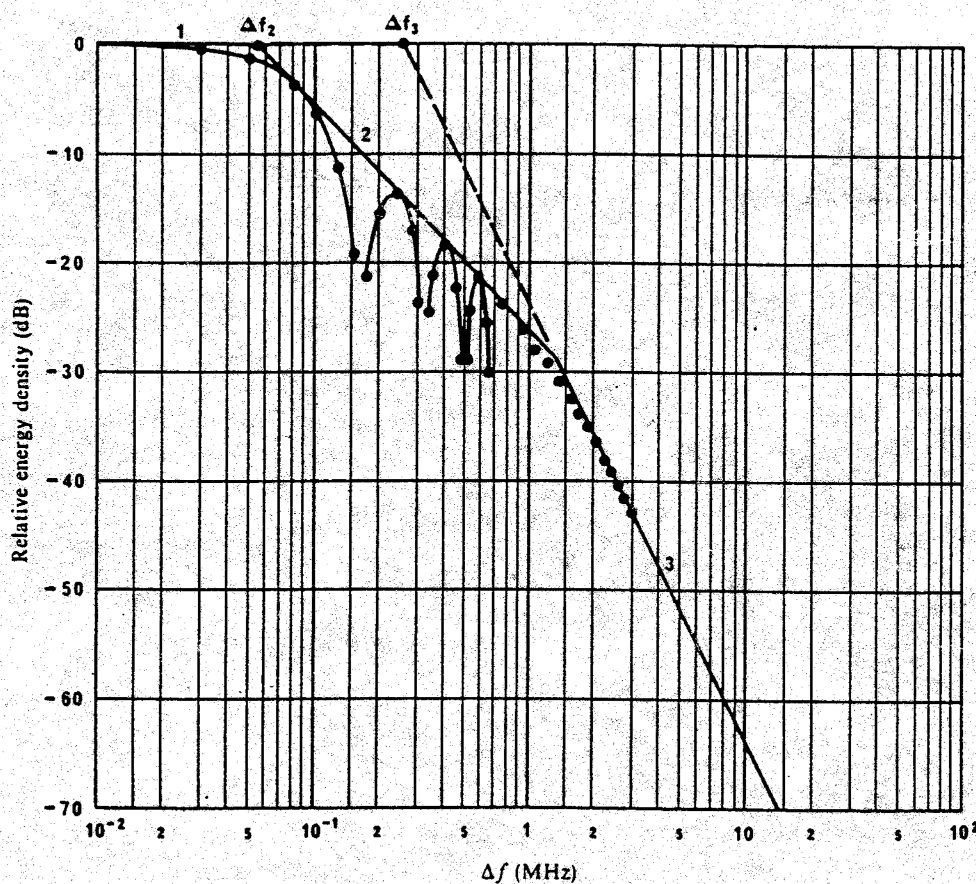


FIGURE 9 — *Relative energy density of a trapezoidal pulse in Example 1*

Example 2 – Linear chirp pulse

To illustrate the procedure for determining the approximate bounds on the spectrum of a typical chirp pulse, the following example is considered where the pulse parameters are:

$$\begin{aligned} \beta &= 1 \times 10^6 \text{ Hz, positive deviation} \\ \tau_b &= 102 \times 10^{-6} \text{ s} \\ \delta_r &= \delta_f = 1 \times 10^{-6} \text{ s} \\ \tau &= 101 \times 10^{-6} \text{ s} \end{aligned}$$

Calculations:

Step 1: Determine if $\beta > 2/(\pi\tau)$ and if the pulse is symmetrical.

$2/(\pi\tau) = 0.0063 \times 10^6$ which is less than β; and
$\delta_r = \delta_f$ and therefore the pulse is symmetrical.

Since the pulse meets these conditions calculate Δf_2, Δf_3, $\Delta f_{a\pm}$, and $\Delta f_{b\pm}$ according to Step 1(A).

Step 1(A):

$$\Delta f_2 = \pm \frac{1}{\pi}[1 \times 10^6 / 102 \times 10^{-6}]^{1/2} = 0.032 \times 10^6 \text{ Hz}$$

$$1/\delta = \frac{1}{2}\left[\frac{1}{10^{-6}} + \frac{1}{10^{-6}}\right] = 10^6 \text{ s}^{-1}$$

$$\Delta f_3 = \pm \frac{1}{\pi}[1 \times 10^6 / 102 \times 10^{-6}]^{1/4} [1 \times 10^6]^{1/2} = 0.10 \times 10^6 \text{ Hz}$$

$$\Delta f_{a\pm} = \frac{10^6}{2}\left[1 - \frac{10^{-6}}{102 \times 10^{-6}}\right] = 0.50 \times 10^6 \text{ Hz}$$

$$\Delta f_{b\pm} = 2\Delta f_{a\pm} = 1 \times 10^6 \text{ Hz}$$

Now plot the spectrum bounds as follows and as shown in Fig. 10.

Step 2: Draw line 1 horizontally through 0 dB.

Step 3: Is $\Delta f_b > 1/(\pi\delta)$?

$1/(\pi\delta) = 10^6/\pi = 0.32 \times 10^6$ Hz which is less than Δf_b. Therefore line 2 is not used in the approximation of the spectrum bounds. Proceed to step 4.

Step 4: Starting on line 1 in Fig. 10, at $\Delta f = \Delta f_3$, draw line 3 with a slope of -40 dB/decade.

Step 5: Locate point *a* 6 dB down at $\Delta f = \Delta f_{a\pm}$.

Step 6: Since $\Delta f_{b\pm} > 1/(\pi\delta)$, for this example, locate point *b* at $\Delta f = \Delta f_{b\pm}$ on line 3.

Step 7: Draw line 4 through points *a* and *b*.

The approximate bounds of the spectrum are formed by the solid lines 1, 4 and 3 on Fig. 10. Line 2 is not used in this particular example because $\Delta f_b > 1/(\pi\delta)$. The pulse is symmetrical therefore, the spectrum is also symmetrical and the plot can be used for either positive or negative values of Δf.

The peak energy density level corresponding to the 0 dB point is calculated as follows:

Assume: peak power (p) = 1×10^6 W.

From equation (8):

$$\begin{aligned} p_d &= [p\tau_b/\beta] \\ &= [(1 \times 10^6)(102 \times 10^{-6})/1 \times 10^6] \\ &= 10.2 \times 10^{-5} \quad \text{Joules/Hz.} \end{aligned}$$

To indicate the accuracy attained in the above example, the spectral power density function, calculated using fast Fourier transform, is also plotted in Fig. 9 using the symbol "·" to designate representative points on the spectral power density function.

The accuracy of bounds obtained with the method described varies over the spectrum, being best for values of Δf of greater than 3β. At this value, the difference between the approximated bound and the theoretical bound is less than 1 dB. As Δf is increased further, the theoretical bound tends to approach line 3 asymptotically and the error rapidly approaches zero. In the region represented by line 1, the error usually is in the order of 1 dB.

The largest error occurs in the region represented by line 4 (Fig. 10). For long, symmetrical pulses, the error here is usually less than 6 dB, but for asymmetrical pulses, especially when τ_b is not much larger than either δ_r or δ_f, the error may be as much as 12 dB. Because line 4 is used to represent only a small part of the overall spectrum, recognition that an error can exist in this region is sufficient for most spectrum analysis.

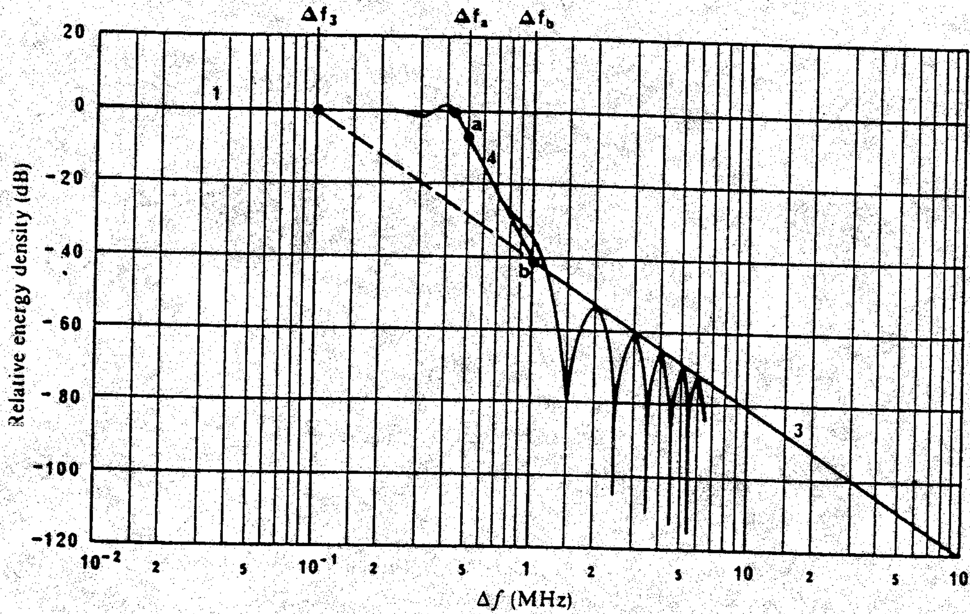


FIGURE 10 — *Relative energy density of a linear FM modulated pulse in Example 2*

4. Conclusions

It is important to recognize that the techniques presented above are first-order approximations of pulse radar emission spectrum shapes. However, the techniques summarized in this Report are of value in most spectrum analysis problems because of the straightforward calculations involved for a reasonably accurate spectrum approximation. Some examples of spectrum analyses to which these techniques would be applicable are: preliminary assessment of radar interference potential as described in Report 654; and radar frequency assignment procedures. These procedures can also be of value in assessing the spectrum requirements for existing radar systems.

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