APPENDIX A: THEORETICAL *PSD* OF DP AND DS SIGNALS

A general expression for the power spectral density (*PSD*) of ultrawideband (UWB) signals using a fixed time-base dither was provided in Section 3 of [2]. This section provides the derivation of the expression and uses it to evaluate the *PSD* of DP and DS signals.

UWB signals that modulate pulses to convey binary data can be expressed as

$$u(t) = \sum_{n=-\infty}^{\infty} \alpha_{0n} y_0(t - nT - \beta_n) + \alpha_{1n} y_1(t - nT - \beta_n) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{1} \alpha_{kn} y_k(t - nT - \beta_n) \quad , \qquad \text{(A-1)}$$

where y_k is the channel symbol, e.g., y_0 represents a zero and y_1 represents a one, T is the nominal symbol period, n is the symbol period index, and β_n is a random variable that defines the dithering as described by the dither function $q(\beta)$. α_{kn} chooses the appropriate symbol for each information bit according to

$$\alpha_{kn} = \begin{cases} 1 - a_n & (k = 0) \\ a_n & (k = 1) \end{cases} \qquad \text{and} \qquad a_n = \begin{cases} 0 & \text{(with probability } \xi_0) \\ 1 & \text{(with probability } \xi_1 = 1 - \xi_0) \end{cases} \quad .$$

This formulation can be altered to map any two distinct information bit values, e.g., {1, -1}, with no change in the final *PSD* expressions. Note that a_n and β_n are independent and identically distributed (iid).

The *PSD* is the Fourier transform of the autocorrelation function, i.e., $R_{uu}(t_1, t_2) = \mathrm{E}\{u(t_1)u^*(t_2)\}$, where the expectation operator, defined as $\mathrm{E}\{X\} = \int xq(x)dx$, is distributive and for independent random variables the expectation of the product is equal to the product of the individual expectations. The complex conjugate * can be dropped because $u(t)$ is a real function, and the autocorrelation function is expressed as

$$R_{uu}(t_1, t_2) = \mathrm{E}\{u(t_1)u(t_2)\} = \mathrm{E}\left\{ \sum_{n,m} \sum_{k,l} \alpha_{kn}\alpha_{lm} y_k(t_1 - nT - \beta_n) y_l(t_2 - mT - \beta_m) \right\} \quad .$$

Note that summations involving n and/or m imply a range $\pm\infty$, summations involving k and/or l imply a range from 0 to 1, and integrals with unspecified limits imply the limits $\pm\infty$.

Evaluation of $R_{uu}(t_1, t_2)$ involves summation of the function

$$\gamma_{nm}(t_1, t_2) = \sum_{k,l} \mathrm{E}\{\alpha_{kn}\alpha_{lm}\} \mathrm{E}\{y_k(t_1 - nT - \beta_n) y_l(t_2 - mT - \beta_m)\} \quad ,$$

where the expectation was moved inside the summation operators and split into the product of expectations of the independent random variables. The expectation of the product of the channel

symbols is difficult to simplify in a meaningful way. To avoid this difficulty, the following variable is introduced

$$\hat{\gamma}_{nm}(t_1,t_2) = \sum_{k,l} \mathrm{E}\{\alpha_{kn}\}\mathrm{E}\{\alpha_{lm}\}\mathrm{E}\{y_k(t_1 - nT - \beta_n)\}\mathrm{E}\{y_l(t_2 - mT - \beta_m)\} \quad .$$

$\sum_{n\neq m}\gamma_{nm}(t_1,t_2) = \sum_{n\neq m}\hat{\gamma}_{nm}(t_1,t_2)$ because the symbols are independent in different symbol periods. However, $\sum_n \gamma_{nn}(t_1,t_2) \neq \sum_n \hat{\gamma}_{nn}(t_1,t_2)$ because the symbols are not independent in the same symbol period. Since both quantities are easily evaluated along the diagonal, $\sum_n \hat{\gamma}_{nn}(t_1,t_2)$ is subtracted and $\sum_n \gamma_{nn}(t_1,t_2)$ is added to formulate the expression in a way that can be evaluated analytically, i.e.,

$$R_{uu}(t_1,t_2) = \sum_{n,m}\gamma_{nm}(t_1,t_2) = \sum_{n,m}\hat{\gamma}_{nm}(t_1,t_2) + \sum_n\left[\gamma_{nn}(t_1,t_2) - \hat{\gamma}_{nn}(t_1,t_2)\right] \quad . \tag{A-2}$$

As will be demonstrated, this approach has the added benefit of distinguishing between the discrete and continuous spectra of the resulting *PSD*.

Consider the first term of equation (A-2), $\sum_{n,m}\hat{\gamma}_{nm}(t_1,t_2)$. $\mathrm{E}\{\alpha_{kn}\}$ is evaluated as

$$\mathrm{E}\{\alpha_{1n}\} = \mathrm{E}\{a_n\} = (0)\xi_0 + (1)\xi_1 = \xi_1$$

$$\mathrm{E}\{\alpha_{0n}\} = \mathrm{E}\{1 - a_n\} = \mathrm{E}\{1\} - \mathrm{E}\{a_n\} = \xi_0 \quad .$$

Hence, $\mathrm{E}\{\alpha_{kn}\} = \xi_k$ is independent of n and can be moved outside $\sum_n$, yielding

$$\sum_{n,m}\hat{\gamma}_{nm}(t_1,t_2) = \left[\sum_k \xi_k \sum_n \mathrm{E}\{y_k(t_1 - nT - \beta_n)\}\right]\left[\sum_l \xi_l \sum_m \mathrm{E}\{y_l(t_2 - mT - \beta_m)\}\right] \quad .$$

Expanding the expectation into integral form and introducing $\phi_k(t) = \sum_n y_k(t - nT)$ gives

$$\sum_n \mathrm{E}\{y_k(t - nT - \beta_n)\} = \sum_n \int y_k(t - nT - \beta)q(\beta)d\beta = \int \phi_k(t-\beta)q(\beta)d\beta = (\phi_k * q)(t) \quad ,$$

where $*$ is the convolutional operator. Defining the function $\psi(t) = \sum_k \xi_k \phi_k(t)$ gives

$$\sum_{n,m}\hat{\gamma}_{nm}(t_1,t_2) = \left[(\psi * q)(t_1)\right]\left[(\psi * q)(t_2)\right] \quad , \tag{A-3}$$

where the periodic function $\psi(t)$ can be expressed as a Fourier series

$$\psi(t) = \sum_m \Psi_m e^{j2\pi mt/T} \quad , \qquad \Psi_m = \frac{1}{T}\int_{-T/2}^{T/2} \psi(t) e^{-j2\pi mt/T} dt \quad .$$

This allows for the convolution in equation (A-3) to be expressed as

$$\begin{aligned}(\psi * q)(t) &= \int \sum_m \Psi_m e^{j2\pi m(t-\beta)/T} q(\beta) d\beta = \sum_m \Psi_m e^{j2\pi mt/T} \int q(\beta) e^{-j2\pi m\beta/T} d\beta \\ &= \sum_m \Psi_m Q\left(\frac{m}{T}\right) e^{j2\pi mt/T} \quad ,\end{aligned}$$

where Q is the Fourier transform of q. The Fourier coefficients in terms of ξ_k and y_k are

$$\Psi_m = \frac{1}{T}\sum_k \xi_k \int y_k(t') e^{-j2\pi mt'/T} dt' = \frac{1}{T}\sum_k \xi_k Y_k\left(\frac{m}{T}\right) \quad ,$$

where change of variable, $t' = t - nT$, was used to simplify and Y_k is the Fourier transform of y_k. Substitution into equation (A-3) and changing the independent variables to $\tau = t_2 - t_1$ and $t = t_2$ yields

$$\sum_{n,m} \hat{\gamma}_{nm}(\tau, t) = \frac{1}{T^2}\sum_{n,m}\left[\sum_k \xi_k Y_k\left(\frac{n}{T}\right)\right]\left[\sum_k \xi_k Y_k\left(\frac{m}{T}\right)\right] Q\left(\frac{n}{T}\right) Q\left(\frac{m}{T}\right) e^{j2\pi[(m+n)t - m\tau]/T} \quad . \qquad \text{(A-4)}$$

The next term in equation (A-2) to be considered is

$$\begin{aligned}\sum_n \hat{\gamma}(t_1, t_2) &= \sum_{k,l} \xi_k \xi_l \sum_n \left(\int y_k(t_1 - nT - \beta) q(\beta) d\beta\right)\left(\int y_l(t_2 - nT - \beta) q(\beta) d\beta\right) \\ &= \sum_{k,l} \xi_k \xi_l \iint Y_k(f_1) Y_l(f_2) Q(f_1) Q(f_2) \left[\sum_n e^{-j2\pi(f_1+f_2)nT}\right] e^{j2\pi(f_1 t_1 + f_2 t_2)} df_1 df_2 \quad ,\end{aligned}$$

where $\int y_k(t - nT - \beta) q(\beta) d\beta = \int Y_k(f) Q(f) e^{-j2\pi fnT} e^{j2\pi ft} df$.

The term in square brackets may be expressed as the sum of delta functions using the identity

$$\sum_n e^{-j2\pi fnT} = \frac{1}{T}\sum_n \delta\left(f - \frac{n}{T}\right) \quad . \qquad \text{(A-5)}$$

Simple evaluation of the f_2 integral, changing the independent variables to $\tau = t_2 - t_1$ and $t = t_2$, and consolidating the k and l terms yields

$$\sum_n \hat{\gamma}_{nn}(t_1,t_2) = \frac{1}{T}\sum_n e^{j2\pi nt/T}\int\left[\sum_k \xi_k Y_k(f)\right]\left[\sum_l \xi_l Y_l\left(\frac{n}{T}-f\right)\right]Q(f)Q\left(\frac{n}{T}-f\right)e^{-j2\pi f\tau}df$$
$$= \frac{1}{T}\sum_n e^{j2\pi nt/T}\left[\left(\sum_k \xi_k y_k(\tau)\right)*\left(\sum_l \xi_l y_l(-\tau)e^{j2\pi n\tau/T}\right)\right]*\left[q(\tau)*\left(q(-\tau)e^{j2\pi n\tau/T}\right)\right] \quad , \tag{A-6}$$

where $\int Y_k\left(\frac{n}{T}-f\right)e^{j2\pi f\tau}df = y_k(-\tau)e^{j2\pi n\tau/T}$.

The last term to be evaluated in equation (A-2) is

$$\sum_n \gamma_{nn}(t_1,t_2) = \sum_n\sum_{k,l} \mathrm{E}\{\alpha_{kn}\alpha_{kn}\}\mathrm{E}\{y_k(t_1-nT-\beta_n)y_l(t_2-nT-\beta_n)\} \quad .$$

$\mathrm{E}\{\alpha_{kn}\alpha_{ln}\}$ is independent of n, that is

$$\mathrm{E}\{\alpha_{1n}\alpha_{1n}\} = \mathrm{E}\{a_n^2\} = (0)^2\xi_0 + (1)^2\xi_1 = \xi_1$$

$$\mathrm{E}\{\alpha_{0n}\alpha_{0n}\} = \mathrm{E}\{(1-a_n)^2\} = \mathrm{E}\{1\} - 2\mathrm{E}\{a_n\} + \mathrm{E}\{a_n^2\} = \xi_0$$

$$\mathrm{E}\{\alpha_{0n}\alpha_{1n}\} = \mathrm{E}\{\nu_{1n}\nu_{0n}\} = \mathrm{E}\{(1-a_n)a_n\} = \mathrm{E}\{a_n\} - \mathrm{E}\{a_n^2\} = 0 \quad .$$

Hence, $\mathrm{E}\{\alpha_{kn}\alpha_{ln}\} = \xi_k\delta_{kl}$, where δ_{kl} is the Kronecker delta function. Expanding the expectation into integral form, taking the double Fourier transform, and consolidating n and β terms yields

$$\sum_n \gamma_{nn}(t_1,t_2) = \sum_k \xi_k \sum_n \int y_k(t_1-nT-\beta)y_k(t_2-nT-\beta)q(\beta)d\beta$$
$$= \sum_k \xi_k \iint Y_k(f_1)Y_k(f_2)\left[\sum_n e^{-j2\pi(f_1+f_2)nT}\right]\left[\int q(\beta)e^{-j2\pi(f_1+f_2)\beta}d\beta\right]e^{j2\pi(f_1t_1+f_2t_2)}df_1df_2 \quad .$$

Utilizing equation (A-5) to evaluate the f_2 integral, changing independent variables to $\tau = t_2 - t_1$ and $t = t_2$, rearranging terms, and recognizing the Fourier transform with respect to τ yields

$$\sum_n \gamma_{nn}(\tau,t) = \frac{1}{T}\sum_n e^{j2\pi nt/T}Q\left(\frac{n}{T}\right)\int\sum_k \xi_k Y_k(f)Y_k\left(\frac{n}{T}-f\right)e^{-j2\pi f\tau}df$$
$$= \frac{1}{T}\sum_n e^{j2\pi nt/T}Q\left(\frac{n}{T}\right)\sum_k \xi_k y_k(\tau)*\left(y_k(-\tau)e^{j2\pi n\tau/T}\right) \quad . \tag{A-7}$$

Equations (A-2), (A-4), (A-6), and (A-7) combine to give an expression for the autocorrelation function in terms of τ and t. The statistics for $R_{uu}(\tau, t)$ are periodic with period T. Such processes are commonly referred to as cyclostationary. For these processes, it is useful to calculate the average over all possible observation times within a period. The time average over one period is denoted as follows

$$\langle R_{uu}(\tau)\rangle = \frac{1}{T}\int_0^T R_{uu}(\tau,t)dt \quad .$$

Evaluating this integral involves the integral identity

$$\frac{1}{T}\int_0^T e^{j2\pi kt/T}dt = \begin{cases} 1 & (k=0) \\ 0 & \text{(otherwise)} \end{cases} \quad ,$$

where k is an integer. Hence, the time average of equation (A-4) is non-zero only for $n = -m$, and the time average of equations (A-6) and (A-7) are non-zero only for $n = 0$. The time-average autocorrelation function reduces to

$$\begin{aligned}\langle R_{uu}(\tau)\rangle = &\frac{1}{T^2}\sum_n \left|\sum_k \xi_k Y_k\left(\frac{n}{T}\right)\right|^2 \left|Q\left(\frac{n}{T}\right)\right|^2 e^{j2\pi n\tau/T} \\ &+\frac{1}{T}\left\{\sum_k \xi_k y_k(\tau) * y_k(-\tau) - \left[\left(\sum_k \xi_k y_k(\tau)\right) * \left(\sum_k \xi_k y_k(-\tau)\right)\right] * \left[q(\tau) * q(-\tau)\right]\right\} \quad ,\end{aligned}$$

where $Q(0)$ is the integrated probability density function $q(\beta)$ equal to 1.

Finally, taking the Fourier transform of $\langle R_{uu}(\tau)\rangle$ yields

$$\begin{aligned}\langle S_{uu}(f)\rangle = \mathrm{FT}\{\langle R_{uu}(\tau)\rangle\} = &\frac{1}{T^2}\left|\sum_k \xi_k Y_k(f)\right|^2 |Q(f)|^2 \sum_n \delta\left(f - \frac{n}{T}\right) \\ &+\frac{1}{T}\left\{\sum_k \xi_k |Y_k(f)|^2 - \left|\sum_k \xi_k Y_k(f)\right|^2 |Q(f)|^2\right\} \quad ,\end{aligned} \tag{A-8}$$

which is comprised of discrete and continuous components that depend on the symbol spectrum and the Fourier transform of the density function used to randomize the signal.

A.1. DP-UWB

For DP signals, $q(\beta)$ is uniformly distributed (and more importantly symmetric); hence, the signal can be expressed according to the general expression (A-1). The DP channel symbol y_1 is a time advanced version of y_0 defined as

$$y_1\left(t+\frac{\beta_{max}T}{2}\right)=y_0(t)\equiv p_{DP}(t) \quad ,$$

where β_{max} is the dither fraction and $p_{DP}(t)$ is the pulse shape. Assuming equiprobable DP information bits, i.e., $\xi_k = ½$, and knowing that time shift translates to a phase shift in the frequency domain results in

$$\sum_k \xi_k \left|Y_k(f)\right|^2 = \frac{1}{2}\left[\left|\int p_{DP}(t)e^{-j2\pi ft}dt\right|^2 + \left|e^{j\pi f\beta_{\max}T}\int p_{DP}(t)e^{-j2\pi ft}dt\right|^2\right] = \left|P_{DP}(f)\right|^2$$

$$\left|\sum_k \xi_k Y_k(f)\right|^2 = \frac{1}{4}\left|P_{DP}(f)\right|^2\left|1+e^{j\pi f\beta_{max}T}\right|^2 = \frac{1}{2}\left|P_{DP}(f)\right|^2\left[1+\cos(\pi f\beta_{max}T)\right] \quad .$$

Substitution in equation (A-8) yields

$$\begin{aligned}\langle S_{uu}(f)\rangle &= \frac{1}{2T^2}\left|P_{DP}(f)Q(f)\right|^2\left[1+\cos\left(\pi f\beta_{max}T\right)\right]\sum_n \delta\left(f-\frac{n}{T}\right)\\ &+\frac{1}{T}\left|P_{DP}(f)\right|^2\left\{1-\frac{\left|Q(f)\right|^2\left[1+\cos\left(\pi f\beta_{max}T\right)\right]}{2}\right\} \quad .\end{aligned}$$

Figures A-1 – A-4 illustrate the power spectral densities of DP-01 – DP-04, respectively. The magnitudes of the spectra were normalized to the peak of the continuous spectrum. For DP-01 – DP-04, $q(\beta)$ is uniformly distributed over half of the pulse period. Consequently, $\beta_{max} = 0.5$ and $Q(f) = \mathrm{sinc}(\pi\beta_{max}Tf)$. This sinc function has nulls at frequencies equal to $2k/T$ ($k = \pm1, \pm2, \ldots$), which zeros out every other spectral line and causes the spacing between discrete spectral lines to be $2/T$. In these plots, the vertical shaded regions in the top plots correspond to the DTV channel displayed in the bottom plots. Notice that in all cases the spectral lines were either below the continuous spectrum or outside the DTV channel.

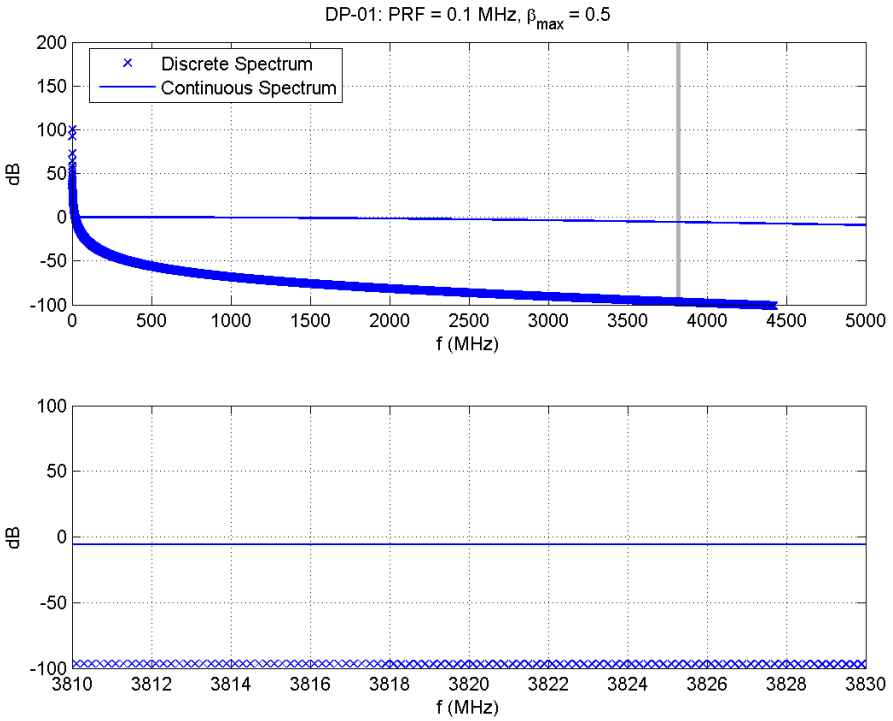

Figure A-1. Theoretical discrete and continuous spectra of DP-01.

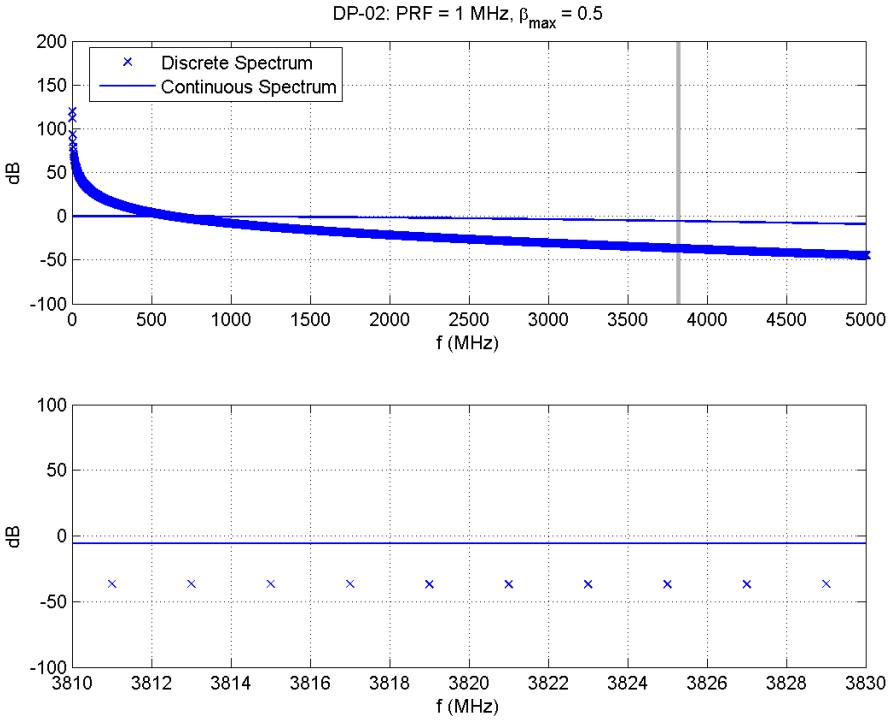

Figure A-2. Theoretical discrete and continuous spectra of DP-02.

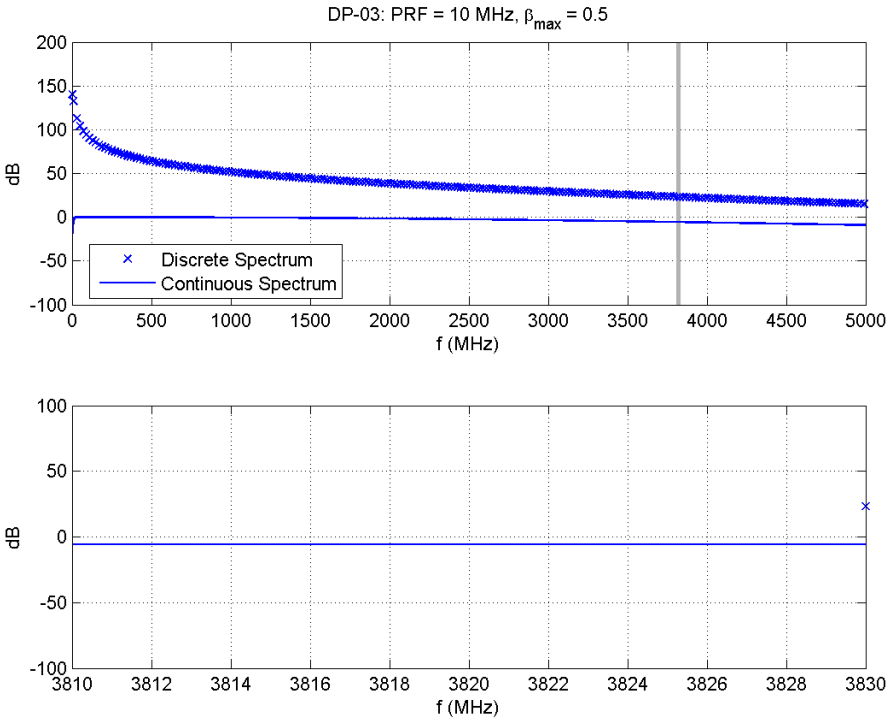

Figure A-3. Theoretical discrete and continuous spectra of DP-03.

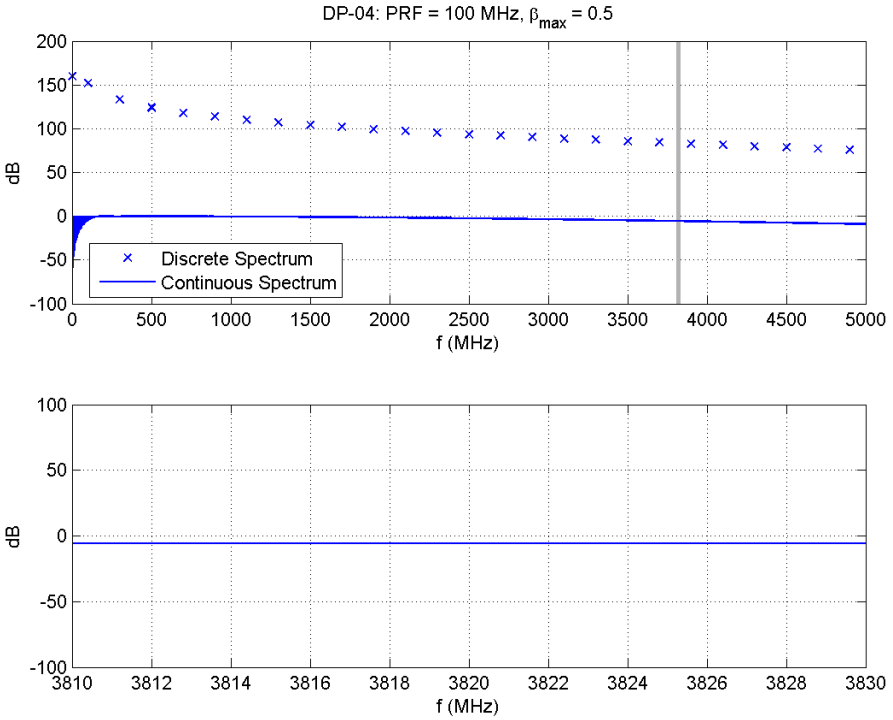

Figure A-4. Theoretical discrete and continuous spectra of DP-04.

A.2. DS-UWB

DS channel symbols are not dithered, i.e., $Q(f) = \mathrm{FT}\{q(\beta)\} = \mathrm{FT}\{\delta(0)\} = 1$. They are defined as the convolution of a codeword and the DS pulse shape (p_{DS}), i.e.,

$$-y_1(t) = y_0(t) \equiv \sum_{l=0}^{L-1} c_l p_{DS}(t - lT_{chip}) \quad ,$$

where L is the codeword length with values of {1, 3, 6, 12, or 24}, c_l is the lth codeword chip with values {-1, 0, 1}, and T_{chip} is the chip period. Assuming DS information bit values are equiprobable, the negative relation between y_1 and y_0 makes

$$\left|\sum_k \xi_k Y_k(f)\right|^2 = \frac{1}{2}\int \left[y_0(t) + (-y_0(t))\right] e^{-j2\pi ft} dt = 0 \quad ,$$

and causes the discrete component of the DS-UWB *PSD* to be zero. The other sum in equation (A-7) evaluates to

$$\begin{aligned}\sum_k \xi_k \left|Y_k(f)\right|^2 &= \left|\sum_{l=0}^{L-1} c_l \int p_{DS}(t - lT_{chip}) e^{-j2\pi ft} dt\right|^2 \\ &= \left|P_{DS}(f)\right|^2 \left|\sum_{l=0}^{L-1} c_l e^{-j2\pi flT_{chip}}\right|^2 \\ &= \left|P_{DS}(f)\right|^2 \left|C(f)\right|^2 \quad ,\end{aligned}$$

and substitution into equation (A-8) yields

$$\left\langle S_{uu}(f)\right\rangle = \frac{1}{T}\left|P_{DS}(f)\right|^2 \left|C(f)\right|^2 \quad .$$

Figures A-5 and A-6 illustrate $|C(f)|$ for DS-04 and DS-06, respectively. In these plots, the magnitudes of the spectra were normalized to their peak, and the vertical shaded regions in the top plots correspond to the DTV channel displayed in the bottom plots. Notice the resemblance between the $|C(f)|$ and the measured *PSD* of DS-06 illustrated in Section 2.3.3. This agreement is because $|P_{DS}(f)|$ is flat over the frequency range 3.7 – 4.2 GHz as illustrated in Figure 3 of Part 1 [6]. Not shown in this appendix are frequency flat sparse codes for DS-01, DS-02, DS-03, and DS-05, which produce the flat measured *PSD*s shown in Section 2.3.3.

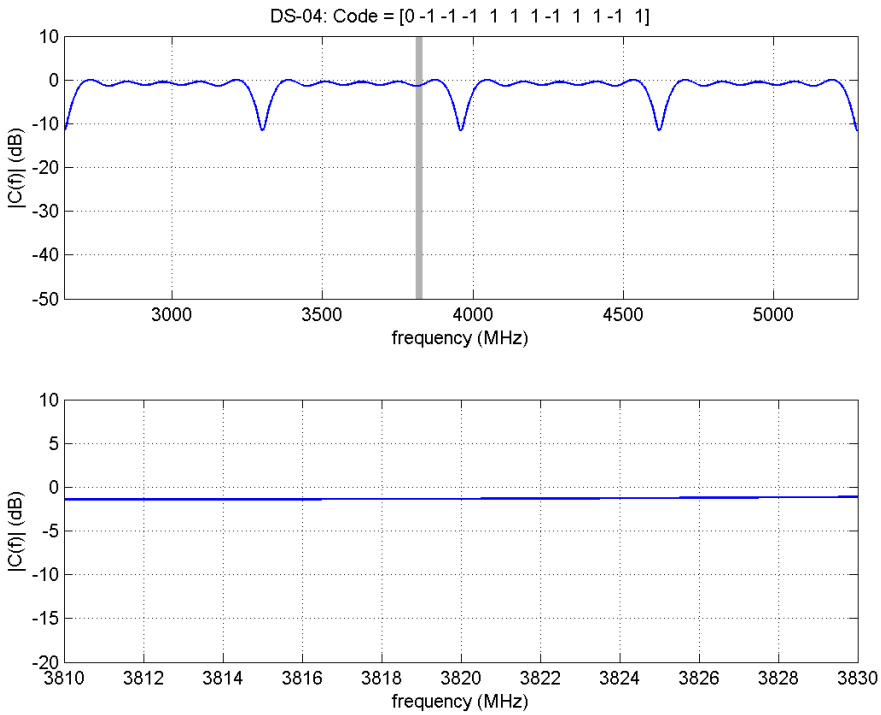

Figure A-5. $|C(f)|$ of the DS-04 code word.

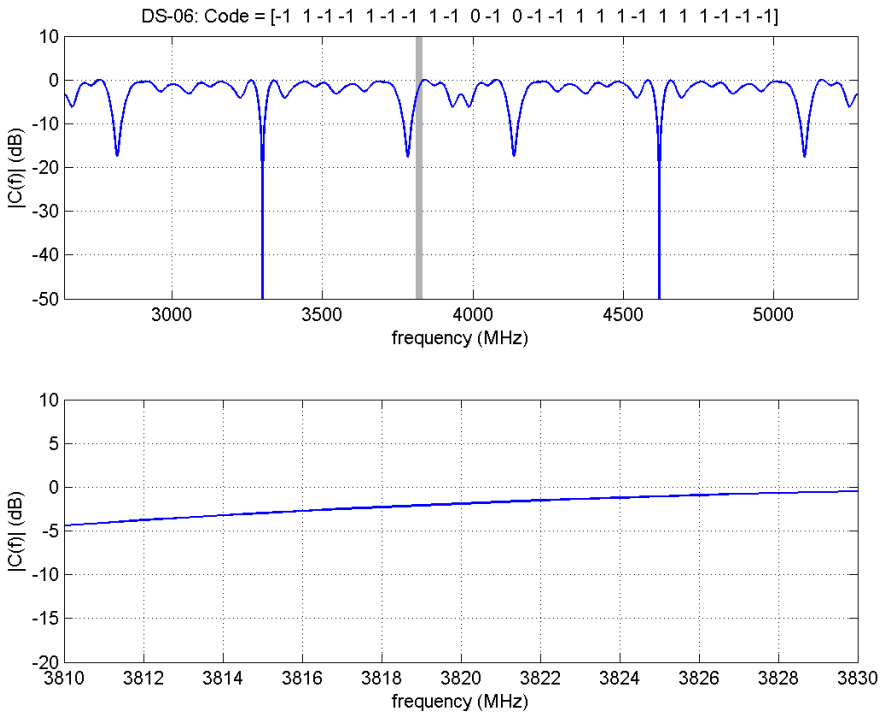

Figure A-6. $|C(f)|$ of the DS-06 code word.