APPENDIX A: DUCTING EXPRESSIONS

Millington [1957] showed that a radio wave trajectory within a layer of constant refractivity gradient may be closely approximated by a parabolic arc. His results can be expressed as

$$y(x) = x\theta_o + x^2 G/2 \qquad \text{m}, \tag{A1}$$

$$\theta(x) = \theta_o + xG \qquad \text{mrad}, \tag{A2}$$

$$G = [g + 157]\, 10^{-3} \qquad \text{M units/m}. \tag{A3}$$

The $0 \leq y \leq \delta h$ of (A1) gives the height of a point on the wave trajectory in meters above the layer base elevation (h_o in Figures 5 and 6) and at a distance of x kilometers from the origin ($y = 0$, $x = 0$). Note that (A1) is relative to the layer base, regardless of its shape; its shape may change as the layer is arched or flat, but the equation is unchanged. Equation (A2) determines the elevation angle along the wave trajectory in milliradians and is the derivative (with respect to x) of (A1) for the small-angle approximation $\theta(x) \simeq \tan \theta(x)$. The layer's modified refractivity gradient $G = G_o < 0$ M units/m would be determined by (A3) from the layer's ducting refractivity gradient $g_o < -157$ N units/km. For that portion of the duct below the ducting layer, then $y(x) = h(x) - h_o \leq 0$, where $h_b \leq h(x) \leq h_o$. The modified refractivity gradient $G = G_b > 0$ M units/m is also determined from (A3), but for a refractivity gradient $g_b > -157$ N units/km. Commonly,

$$G_b = [-40 + 157]\, 10^{-3} = 0.117 \quad \text{M units/m}. \tag{A4}$$

By manipulation of (A1), (A2), and (A3), several of the trajectory characteristics may be determined. For example, the maximum take-off angle (at $x = o$) for a trapped trajectory is known as the critical take-off angle

$$\theta_c = \pm \sqrt{2|\delta M|} \qquad \text{mrad}, \tag{A5}$$

where the choice of sign is such as to maintain θ_c/G positive. In (A5),

$$\delta M = G_o\, \delta h < 0 \qquad \text{M units}, \tag{A6}$$

and δh is the ducting-layer thickness in meters. The duct thickness is, from (6),

$$D = \delta h \, [1 - G_o/G_b] \qquad \text{m.} \tag{A7}$$

For an initial elevation angle θ_o at $y = 0$, the wave trajectory parabolic arc within the layer will have, from (A2), a maximum elevation $\hat{y}$ at

$$\hat{x} = -\,\theta_o/G_o \qquad \text{km,} \tag{A8}$$

and

$$\hat{y} = \theta_o \hat{x}/2 = -\theta_o^2/2G_o \qquad \text{m.} \tag{A9}$$

Of course,

$$\frac{\hat{y}}{\delta h} = \left(\frac{\theta_o}{\theta_c}\right)^2 \leq 1.0 \tag{A10}$$

At the point $y(x)$,

$$x = \hat{x} \, [1 \mp Q(x)] \qquad \text{km,} \tag{A11}$$

and

$$\theta(x) = \pm \, \theta_o \, Q(x) \qquad \text{mrad,} \tag{A12}$$

where the choice of sign is the opposite of that for (A11). For example, $\theta(x) > 0$ for $x < \hat{x}$. The $Q(x)$ is given by

$$Q(x) = \sqrt{1 - y(x)/\hat{y}} \qquad \text{for } y > 0, \tag{A13}$$

and

$$Q(x) = \sqrt{1-y(x)/\hat{y}_b} \qquad \text{for } y < 0, \tag{A14}$$

where

$$\hat{y}_b = \delta h \frac{G_o}{G_b} < 0 \qquad \text{m.} \tag{A15}$$

For the trajectory parabolic arc, $y(x) \geq 0$, the chord length from $y(x=0)$ to $y(x=x_{max}) = 0$ is

$$(x_{max})_o = 2\hat{x} = -2\theta_o/G_o > 0 \qquad \text{km.} \tag{A16}$$

For the trajectory parabolic arc $y(x) \leq 0$, the chord length is given by

$$(x_{max})_b = - (x_{max})_o \frac{G_o}{G_b} > 0 \qquad \text{km,} \tag{A17a}$$

$$= 2\hat{x}_b = -2\theta_o/G_b > 0 \qquad \text{km.} \tag{A17b}$$