Finally, white Gaussian noise is a special case, and the performance of digital systems in white Gaussian noise has been treated in great detail. Here, we will occasionally refer to the well-known results of system performance in white Gaussian noise for comparison. The pdf for the instantaneous amplitude for Gaussian noise (mean noise power = 1) is:

$$p_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad , \tag{7}$$

and for the corresponding envelope

$$p_R(r) = re^{-r^2/2} \quad , \text{ and} \tag{8}$$

$$P_R(r) = e^{-r^2/2} \quad . \tag{9}$$

3. SYSTEM PERFORMANCE CALCULATIONS

In this Section we want to present the results which are most advantageous for our use. We want to develop system performance algorithms which do not require particularly sophisticated numerical analysis techniques and which can be used on small scale computers.

We start with the simplest. For arbitrary additive interference which is independent from an integration period (bit length) to the next and which has uniformly distributed phase, Montgomery (1954) has shown that the probability of binary bit error, P_e, for NCFSK (non-coherent frequency shift keying) is given by:

$$P_e = \frac{1}{2} \text{ Prob [noise envelope > rms signal level].} \tag{10}$$

While Montgomery's result for NCFSK is in terms of the noise and signal envelopes at the input to an ideal discriminator, it has been shown (White, 1966) that the result is also applicable to most common FSK receivers (bandpass-filter discriminator receivers and matched-filter envelope detection receivers). Using (10), then, the performance for NCFSK can be obtained instantly from the APD measurements, once the APD has been normalized to its rms level. For example, if the two NCFSK waveforms are given by

$$S_1(t) = \sqrt{2S} \cos(\omega_1 t + \phi), \text{ and}$$
$$S_2(t) = \sqrt{2S} \cos(\omega_2 t + \phi), \tag{11}$$

where ϕ is the unknown (uniformly distributed) phase (i.e., incoherent signaling), ω_1 and ω_2 are the two frequencies, and S is the signal power, using (9) and (10), performance in Gaussian noise is, therefore,

$$P_e = \frac{1}{2} e^{-S/2} \quad . \tag{12}$$

For Class A noise, using (4) and (10),

$$P_e = \frac{1}{2} e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} e^{-S/2\sigma_m^2} \quad . \tag{13}$$

The above, of course, is for the binary symmetric channel. That is, $S_1(t)$ and $S_2(t)$ are equally probable. In short, the performance can be obtained by inspection from the normalized APD. If, for example, the probability that R_0=1 (0 dB) is exceeded is P_0, then, for the signaling set given by (11), p_e for a SNR of 2(3 db) is $P_0/2$, and so on, for any SNR. Note the 3 dB "shift" for NCFSK. No algorithm is given in the Appendix for NCFSK, since all that is required is a normalized APD, and the normalization procedure is included in other system performance algorithms. Also, in any case, the APD measurement device, DM-4, would normally present the measurements in normalized form, although we do not make that assumption in this report in order to maintain as much generality as possible.

In the bi-phase, DCPSK (differentially coherent phase shift keying) system, the receiver compares the phase ϕ of a noisy signal with a reference phase $\bar{\phi}$, to decide whether the corresponding pure signal relative phase ψ was 0 or π (ψ = 0, corresponding to the signal $\sqrt{2S}$ cos $\omega_0 t$, is selected if $|\phi-\bar{\phi}|<\pi/2$, and $\psi=\pi$, corresponding to $-\sqrt{2S}$ cos $(\omega_0 t)$, otherwise.) The reference phase is obtained from the previously received signals; usually it is just the phase of the previous signal. Thus the analysis of this system is complicated by the fact that both

ϕ and $\bar{\phi}$ are affected by noise. This system also has adjacent symbol dependency, and, therefore, the occurrence of paired errors and other error groupings cannot be obtained easily, even with independent noise. Halton and Spaulding (1966) have given results for this system, including the occurrence of various error groupings. However, it can be shown that for binary DCPSK, the elemental probability of error, P_e, is the same as for NCFSK, with 3 dB less signal energy required. That is, for a given P_e, DCPSK requires 3 dB less SNR than does NCFSK for arbitrary additive interference that is independent from one bit time to the next. [For a geometrical derivation of this result see Arthurs and Dym (1962).] For example, therefore, for Gaussian noise for binary DCPSK;

$$P_e = \frac{1}{2} e^{-S}. \tag{14}$$

The performance of DCPSK can be obtained directly from the APD of the additive interference. If, for example, the probability that $R_0 = 1$ (0 dB) is exceeded is P_0, then for the above signaling set, P_e for a SNR of 1 (0 dB), is $P_0/2$, and so on for any SNR. Figure 3 shows P_e versus SNR for the noise of figure 1 for both NCFSK and DCPSK, while Figure 4 shows P_e versus SNR for the Class B noise of Figure 2 for these two systems. Performance for Gaussian noise is also shown for reference.

We next consider coherent binary systems. The performance of these systems can be obtained from the pdf of the additive interference envelope by means of the result:

$$P_e = \frac{1}{\pi} \int_K^\infty p_R(r) \cos^{-1}\left(\frac{K}{r}\right) dr . \tag{15}$$

For the derivation of this result see Spaulding (1964), and for various other approaches which led to (15), see Arthurs and Dym (1962). For antipotal signaling (CPSK, coherent phase shift keying), the binary signal set is,

$$\begin{aligned} S_1(t) &= \sqrt{2S} \cos (\omega_0 t), \text{ and} \\ S_2(t) &= -\sqrt{2S} \cos (\omega_0 t), \end{aligned} \tag{16}$$

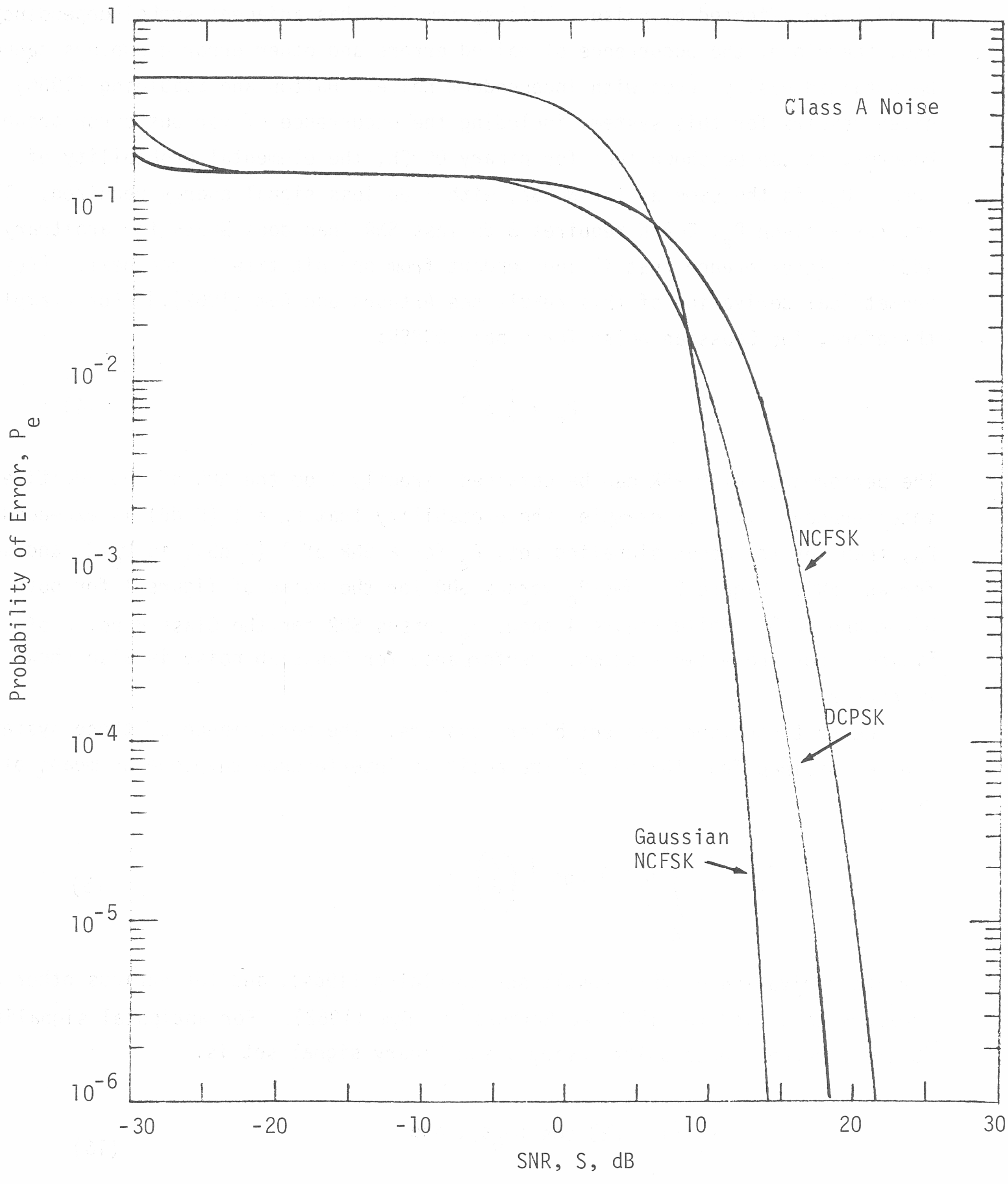

Figure 3. Systems performance for NCFSK and DCPSK for the Class A noise example of Figure 1.

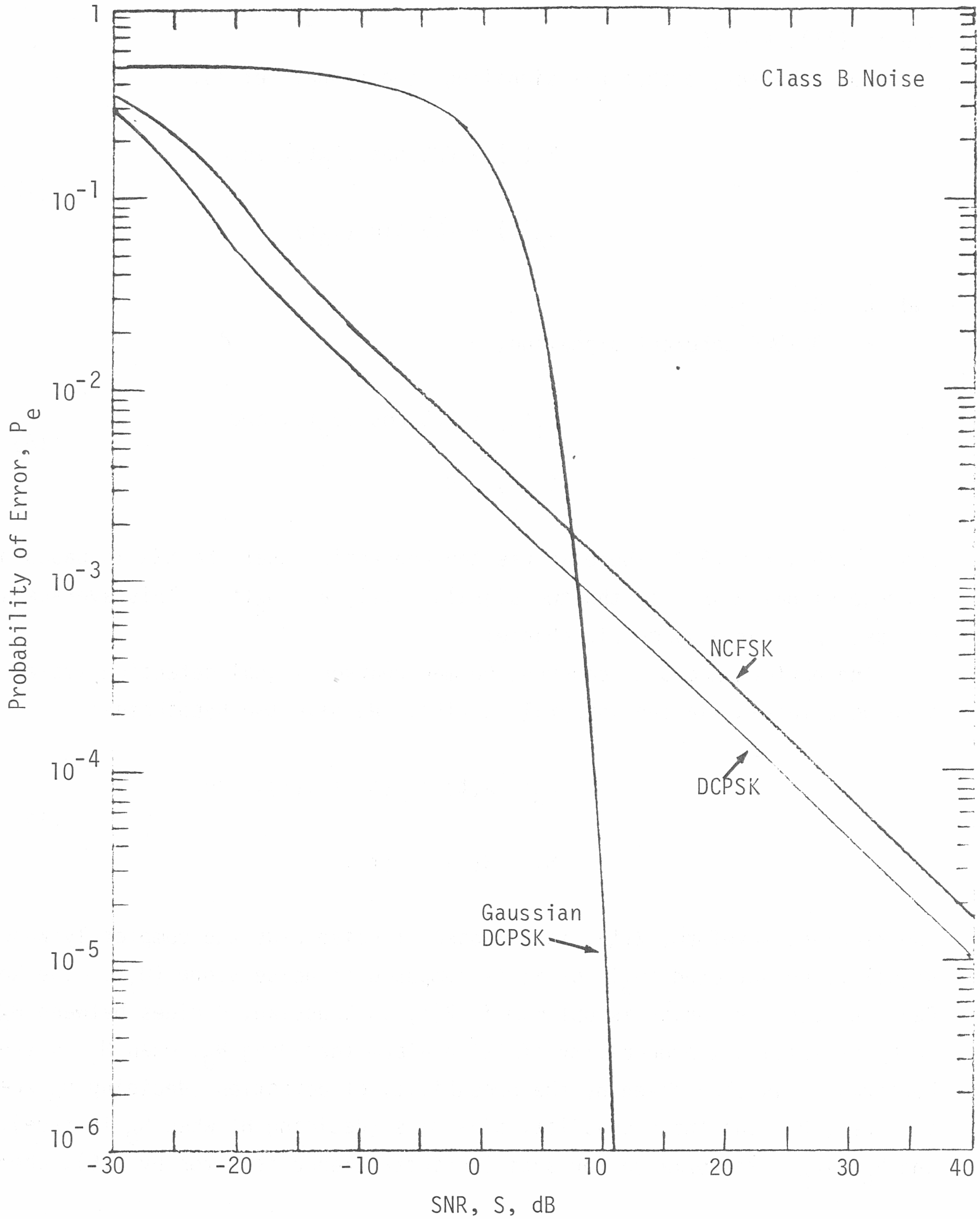

Figure 4. System performance for NCFSK and DCPSK for the Class B noise example of Figure 2.

and in (15), $K = \sqrt{S}$.

For coherent, orthogonal signaling, the signal set is,

$$S_1(t) = \sqrt{2S} \cos (\omega_0 t), \text{ and}$$
$$S_2(t) = \sqrt{2S} \sin (\omega_0 t) \quad , \tag{17}$$

and in (15), $K = \sqrt{S/2}$.

For ON-OFF coherent signaling,

$$S_1(t) = \sqrt{2S} \cos (\omega_0 t), \text{ and}$$
$$S_2(t) = 0 \quad , \tag{18}$$

and in (15), $K = \sqrt{S/4}$, where we use the convention that the SNR is based on the average signal power of the two signals, $S_1(t)$ and $S_2(t)$. This average is, of course, $S/2$, for a symmetric channel.

The performance of a coherent Neyman-Pearson signal detection system can also be obtained via the integral in (15). We have two hypotheses:

$$H_0: \quad X(t) = Z(t) + S(t), \text{ and}$$
$$H_1: \quad X(t) = Z(t). \tag{19}$$

The received waveform, $X(t)$, is composed of noise plus the completely known signal to be detected (H_0), or it is composed of noise alone (H_1). The Neyman-Pearson detector, which is optimum if $Z(t)$ is Gaussian, decides between H_0 and H_1 by presetting a probability of false alarm (deciding H_0, when H_1 is true) α. Performance is then given by the probability of detection (deciding H_0 when H_0 is true), P_D, and the probability of a miss (deciding H_1 when H_0 is true), P_M, and $P_D = 1 - P_M$. Performance for additive interference is given by (15), where

$$K = \sqrt{S} - \sqrt{2}\, \text{erfc}^{-1}(2\alpha), \tag{20}$$

for a desired signal of power S. The complimentary error function is given by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \tag{21}$$

Use of the K given in (20) in (15) gives the probability of a miss, P_M. This is

$$P_M = \frac{1}{\pi} \int_K^{\infty} p_R(r) \cos^{-1}\left(\frac{K}{r}\right) dr, \tag{22}$$

and $P_D = 1 - P_M$, K given by (20).

Table 2 below gives $\mathrm{erfc}^{-1}(2\alpha)$ for various probabilities of false alarm, α.

Table 2. $\mathrm{Erfc}^{-1}(2\alpha)$ for Various α

α	$\mathrm{Erfc}^{-1}(2\alpha)$
10^{-2}	1.645
10^{-3}	2.185
10^{-4}	2.630
10^{-5}	3.015

For the above coherent systems in Gaussian noise,

$$P_e = \frac{1}{2} \mathrm{erfc}(K), \tag{23}$$

where $K = \sqrt{S}$ for antipodal signaling, $K = \sqrt{S/2}$ orthogonal signaling, and $K = \sqrt{S/4}$ for ON-OFF signaling. For the signal detection system in Gaussian noise,

$$P_M = \frac{1}{2} \mathrm{erfc}(K), \tag{24}$$

where $K = \sqrt{S} - \sqrt{2}\, \mathrm{erfc}^{-1}(2\alpha)$.

Likewise, for Class A noise, we can obtain,

$$P_e = \frac{e^{-A}}{2} \sum_{m=0}^{\infty} \frac{A^m}{m!} \operatorname{erfc}(K/\sigma_m) \quad . \tag{25}$$

Equation (25) gives the P_m when the K given by (20) is used.

It now remains to develop efficient computer algorithms based on (15). The result (15) uses the pdf of the interference envelope, and the measurements are of the APD. Actually, the measurements at 31 levels of the APD also give an equally valid estimate of the pdf as well. Also if we attempt to modify (15) to a form that uses the APD directly, i.e., uses $P_R(r)$ rather than $p_R(r)$, we obtain computational complexities. For example, (15) can be transformed to:

$$P_e = \frac{1}{\pi} \int_K^{\infty} P_R(r) \frac{K}{r^2\sqrt{1-K^2/r^2}} \, dr, \tag{26}$$

or

$$P_e = \frac{1}{\pi} \int_0^1 P_R\left(\frac{K}{r}\right) \frac{1}{\sqrt{1-r^2}} \, dr. \tag{27}$$

Both (26) and (27) are improper integrals, and while this creates no problem analytically, very sophisticated numerical integration routines are required in order to obtain any accuracy for P_e, especially when $P_R(r)$ is given only in sampled data form. It turns out that it is much better to use (15) "directly" along with $p_R(r)$ estimated from the measured $P_R(r)$.

The main algorithm presented in the Appendix is called SYSAPD. This program takes the 31 measured APD data points, normalizes the APD to its rms level, obtains the pdf, and then evaluates the integral (15) for the appropriate K. The program SYSAPD uses Gauss-Laguerre quadratures to evaluate (15) (Kopal, 1961). The Gauss-Laguerre quadrature formula is

$$\int_0^{\infty} e^{-x} f(x)dx \quad \sum_{j=1}^{n} H_j f(z_j) \ , \tag{28}$$

the points at which the integrand must be evaluated, y_j, and the corresponding weight, H_j, are obtained via the Laguerre polynominals. The program SYSAPD used a fifteenth order quadrature [(n=15 in (28)]. The above means that the integral (15) is put in the form

$$P_e = \frac{1}{\pi}\int_0^{\infty} \left[p_R(y+K)\, e^y \, \mathrm{Cos}^{-1}\left(\frac{K}{y+K}\right)\right] e^{-y} dy \ , \tag{29}$$

for evaluation.

Consider first the Class A "measurement" data of Table 1. Table 3 gives P_e versus SNR for CPSK obtained from the program SYSAPD which uses (29). Note that in using the data (see program listing in the Appendix) arbitrary 3 dB levels are used. For Class A noise, (25) gives the "correct" theoretical performance. The program SYSCOR computes (25) and Table 3 also gives these results so that the approximation from the "measurements" can be compared with the "true" answer. Another program that is given in the Appendix is SYSGL. This program uses (29), but $p_R(r)$ is obtained from the Class A model mathematical expression (4) rather than from the corresponding "measurement" data. The P_e versus SNR for CPSK from this program is also given on Table 3. This shows the accuracy of the integration routine when these results are compared with the "true" results. It also indicates the accuracy of the normalization, pdf determination, and interpolation techniques used in SYSAPD. Finally, Figure 5 shows the results of Table 3 along with the standard performance in Gaussian noise (23) for further comparison.

The above results are for the Class A example. For the Class B case, the simple Gauss-Laguerre quadrature used above does not give sufficient accuracy when the Class B "measurements" are used in the program SYSAPD or when the corresponding mathematical model for Class B noise is used with program SYSGL. [The result of using the Class B example in SYSAPD (or in SYSGL) is shown by the dashed curve on Figure 6.] Because of this, a different integration routine must be used. This is given by program SYSWR, which used Weddle's Rule (Kopal, 1961) to perform the integrations. This integration routine uses (15) directly and, of course, is somewhat more sophisticated than the Gauss-Laguerre quadrature used previously, but it is still appropriate for small scale computers. For the Class B case, we have no "theoretical" results to use to check the accuracy of the integrations performed by SYSWP.

Table 3. CPSK System Performance for Class A Noise

SNR (dB)	p_e, SYSCOR	p_e, SYSGL	p_e, SYSAPD
-3.00000E+01	1.60710E-01	1.44710E-01	1.39511E-01
-2.75000E+01	1.46402E-01	1.43726E-01	1.38669E-01
-2.50000E+01	1.42528E-01	1.42416E-01	1.37559E-01
-2.25000E+01	1.40644E-01	1.40669E-01	1.36052E-01
-2.00000E+01	1.38309E-01	1.38342E-01	1.34258E-01
-1.75000E+01	1.35202E-01	1.35244E-01	1.30981E-01
-1.50000E+01	1.31073E-01	1.31125E-01	1.26556E-01
-1.25000E+01	1.25602E-01	1.25664E-01	1.20784E-01
-1.00000E+01	1.18386E-01	1.18458E-01	1.13917E-01
-7.50000E+00	1.08950E-01	1.09034E-01	1.04748E-01
-5.00000E+00	9.67994E-02	9.68926E-02	9.24013E-02
-2.50000E+00	8.15682E-02	8.16687E-02	7.69463E-02
0.	6.33627E-02	6.34648E-02	5.82815E-02
2.50000E+00	4.33409E-02	4.34341E-02	4.15365E-02
5.00000E+00	2.42702E-02	2.43404E-02	2.21766E-02
7.50000E+00	9.99877E-03	1.00367E-02	9.42682E-03
1.00000E+01	2.69063E-03	2.70273E-03	2.68102E-03
1.25000E+01	4.46870E-04	4.48876E-04	4.70352E-04
1.50000E+01	4.05121E-05	4.07002E-05	4.63872E-05
1.75000E+01	1.39231E-06	1.39908E-06	2.07974E-06
2.00000E+01	1.32516E-08	1.33143E-08	5.27832E-09
2.25000E+01		2.09713E-11	0.

Table 4. CSPK System Performance for Class B Noise

SNR (dB)	p_e, SYSWR Model	p_e, SYSWR Data
-30.0	1.69496E-01	1.73082E-01
-25.0	7.04404E-02	7.50442E-02
-20.0	2.90375E-02	3.07851E-02
-15.0	1.36975E-02	1.43575E-02
-10.0	6.72071E-03	7.02565E-03
-5.0	3.34009E-03	3.48668E-03
0.0	1.66805E-03	1.73837E-03
5.0	8.34675E-04	8.67205E-04
10.0	4.18022E-04	4.33207E-04
15.0	2.09434E-04	2.15463E-04
20.0	1.04948E-04	1.06017E-04
25.0	5.25944E-05	5.07207E-05
30.0	2.63586E-05	2.44030E-05
35.0	1.32103E-05	1.06807E-05
40.0	6.62078E-06	3.62328E-06

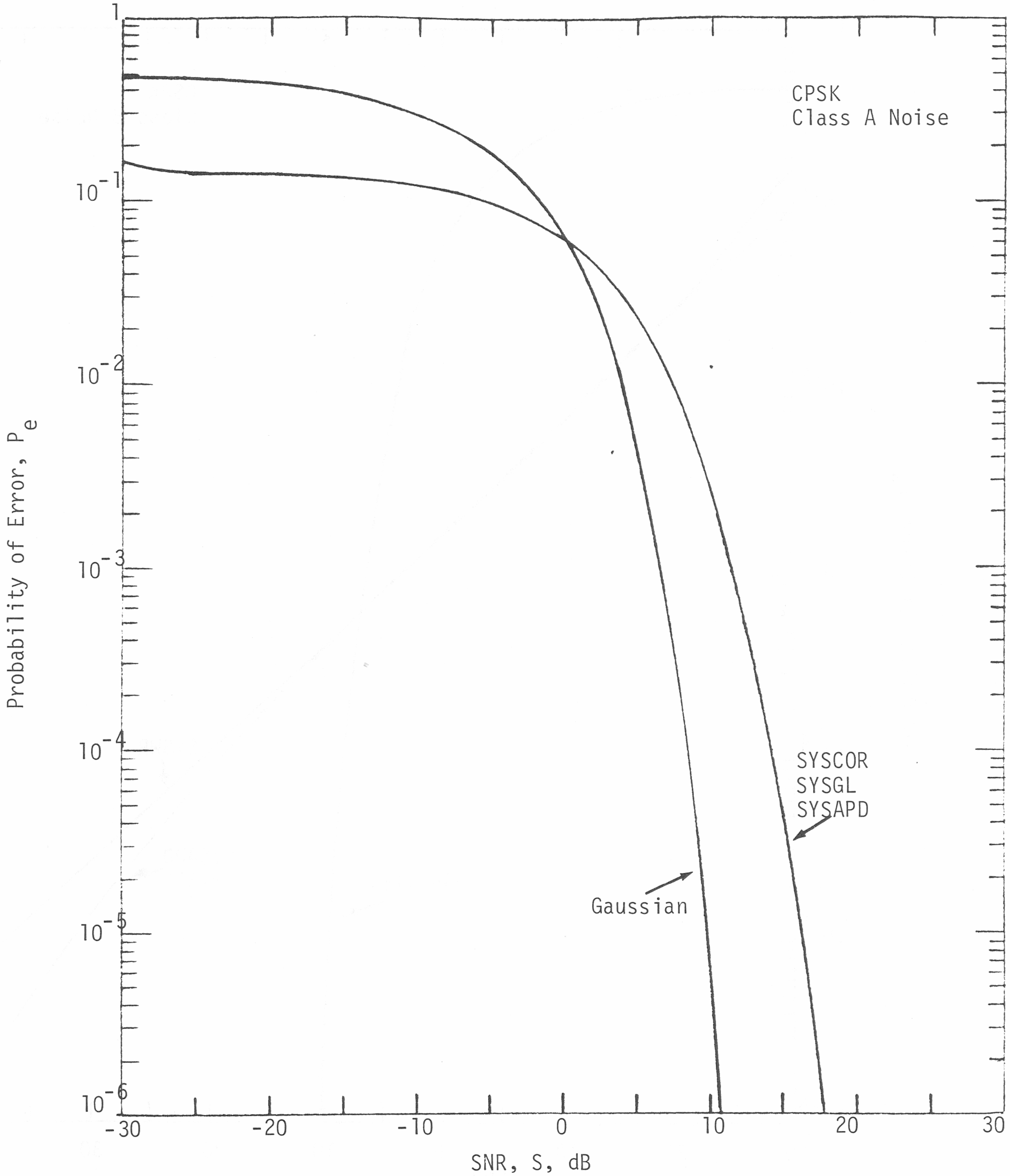

Figure 5. System performance for CPSK for the Class A noise example of Figure 1.

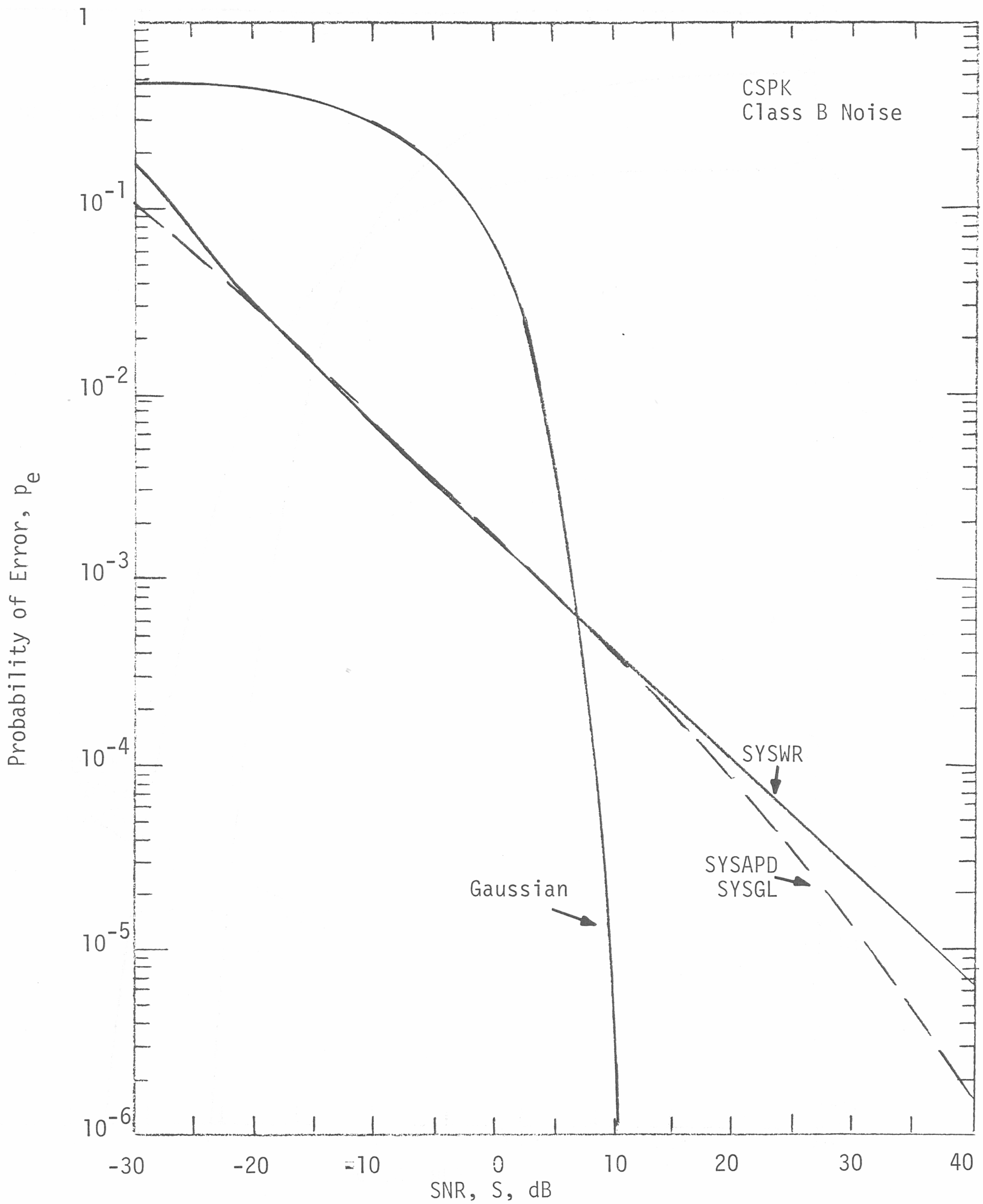

Figure 6. System performance for the Class B noise example of Figure 2.

The accuracy was checked by using another integration routine, appropriate only for large computers, in which the desired accuracy can be specified in conjunction with the Class B mathematical model. Program SYSWP was found to give very good accuracy for all signal-to-noise ratios. Table 4 and Figure 6 show the results of the use of SYSWR with the "measurements" of Table 1. Two subroutines, both termed FUN1, are given. One for the "measurement" data and program SYSWR and one for the mathematical Class B model for use with SYSWR.

In order to evaluate the integral (15) the pdf of the noise envelope is very easily obtained from the APD for Class A noise given in (4). However, obtaining the pdf for the envelope of Class B noise corresponding to the APD given by (6) is somewhat more involved. By differentiating (6) we eventually obtain the following, which has been put in a form suitable for numerical computation:

$$P_R(r) = \frac{2r}{\Omega} e^{-r^2/\Omega} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n A_\alpha^n}{n!} \Gamma\left(1 + \frac{n\alpha}{2}\right) \left[{}_1F_1\left(1-\frac{n\alpha}{2} ; 2; \frac{r^2}{\Omega}\right) - \frac{r^2}{2\Omega}\left(1 + \frac{n\alpha}{2}\right) {}_1F_1\left(1-\frac{n\alpha}{2} ; 3; \frac{r^2}{\Omega}\right)\right]\right\} . \tag{30}$$

The Appendix lists the appropriate programs and all the required subroutines used in the above example calculations.

4. CONCLUSIONS

This report has developed simple computer algorithms which use measurements of the APD of an interfering waveform (or a corresponding mathematical model) to determine performance of various "normal" digital data systems. As can be seen from the examples above, we obtain very good estimates of system performance using SYSAPD and DM-4 Class A simulated noise measurements and by using SYSWR and DM-4 Class B simulated noise measurements. Of course, the Class A measurements can also be used with SYSWR. The algorithms developed are for binary digital systems (and the coherent Neyman-Person signal detection system), however, the performance of other systems (e.g., M level systems, and minimal shift keying systems) can usually be obtained by appropriate extensions of the techniques developed here.