LOCALLY OPTIMUM AND SUBOPTIMUM DETECTOR PERFORMANCE IN A NON-GAUSSIAN INTERFERENCE ENVIRONMENT

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Since the normally assumed white Gaussian interference is the most destructive in terms of minimizing channel capacity, substantial improvement can usually be obtained if the real-world interference environment (non-Gaussian) is properly taken into account. In this report, the performance of the locally optimum Bayes detector (LOBD) is compared with the performance of various ad hoc nonlinear detection schemes. The known results are reviewed and then it is demonstrated that these theoretical results may be misleading due to the assumptions that are required in order to derive them analytically. For a particular type of broadband impulsive noise, the critical assumptions of "sufficiently" small signal level and large number of samples (large time-bandwidth product so that the Central Limit Theorem applies) are removed; the first, analytically, and the second, by computer simulation. The thus derived performance characteristics are then compared, especially as the signal level increases. One result is that there are situations where the bandpass limiter outperforms the LOBD as the signal level increases; that is, the locally optimum detector may not remain "near optimum" in actual operational situations.

Key words: optimum detection; non-Gaussian noise; communication system simulation; parametric signal detection; Class A, B noise

1. INTRODUCTION

The real-world noise environment is almost never Gaussian in character, yet receiving systems in general use are those which are optimum for white Gaussian noise (i.e., linear matched filter or correlation detectors).

It is well known that Gaussian noise is the worst kind of noise in terms of minimizing channel capacity or in its information destroying capability. This means that very large improvements in the performance of systems can be achieved if the actual statistical characteristics of the noise and interference are properly taken into account, and there have been various significant efforts in the last few years in this area (Spaulding and Middleton, 1977; Middleton and Spaulding, 1983).

When confronted with real-world noise, the earlier and usual approach was to precede the "Gaussian receiver" by various ad hoc nonlinearities (e.g., clipper, hole punchers, hard limiters, etc.) in order to make the noise look "more Gaussian" to the given receiver. Later, optimum systems were derived (e.g., Spaulding and Middleton, 1977; Hall, 1966) using models of the actual noise. These systems are

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adaptive in nature and usually very difficult to realize physically. If, however, the following two assumptions are made:

1) the desired signal becomes "sufficiently" small ["sufficiently small" is defined in Middleton and Spaulding (1983)*], and
2) the time-bandwidth product is large, so that a large number, N, of independent samples from the interfering noise process can be used in the detection decision process,

then a "locally optimum" detector, generally termed a "locally optimum Bayes detector" or LOBD, can be obtained. Under some rather strict conditions, these LOBD detectors approach true optimality (asymptotically) as the above two assumptions are met, and usually take the form of the "normal" Gaussian receiver preceded by one or more particular nonlinearities.

In this report, we want to briefly review the derivation of the LOBD, primarily for the case of binary coherent phase shift keying, CPSK, and then review the comparison of the LOBD performance with the performance of the hard-limiter (or other nonlinearity) performance. We do this to point out and have available the results we need to refer to later. In actual use, the desired signal may be "small," but not "small enough," and/or the time bandwidth product may not be particularly large. One of the main objectives, then, of this report is to remove the above two assumptions to investigate the "truth" of the standard LOBD and hard-limiter performance estimates. This is done for one typical example of broadband impulsive noise. For this example case, the first assumption (sufficiently small signal) is removed analytically and the second (large N so that Central Limit Theorem arguments can be used) is removed by computer simulation. Another main objective of this report is to summarize the results from an extensive set of Monte Carlo computer simulation results for the CPSK system, using various nonlinearities (including the LOBD nonlinearity) and also using Rayleigh fading signals as well as constant signals. We start in the next section by reviewing the pertinent standard analytical results for the LOBD and then proceed to remove the assumptions used to obtain the "standard" performance estimates. An appendix contains the computer algorithms used to obtain the Monte Carlo simulation results.

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*See Sections 2.4, 6.4, and Appendix A.3 of Middleton and Spaulding (1983).

2. LOCALLY OPTIMUM DETECTION

The techniques for deriving the locally optimum detector for various signaling situations are well known and covered in detail in Spaulding and Middleton (1977) and the references therein. Here, we simply review the results in order to indicate where the two assumptions above come into play. Our problem, for binary CPSK, is to decide optimally between the two hypotheses:

$$\begin{aligned} H_1 &: \; X(t) = S_1(t) + Z(t) \qquad 0 \le t < T \\ H_2 &: \; X(t) = S_2(t) + Z(t) \qquad 0 \le t < T. \end{aligned} \tag{1}$$

In (1), $X(t)$ is our received waveform in detection time T and this waveform contains either the completely known signal $S_1(t)$ plus the noise $Z(t)$ or the completely known, equi-probable, signal $S_2(t)$ plus $Z(t)$. To obtain our receiver structure we follow the standard procedure of replacing all waveforms by vectors of N samples from the waveforms ($X(t) \rightarrow \underline{X} = \{x_i\}$, etc.) and forming the likelihood ratio $\Lambda(\underline{X})$:

$$\Lambda(\underline{X}) = \frac{p(\underline{X}|H_2)}{p(\underline{H}|H_1)} = \frac{p_Z(\underline{X} - \underline{S}_2)}{p_Z(\underline{X} - \underline{S}_1)} \underset{H_2}{\overset{H_1}{\lessgtr}} 1. \tag{2}$$

When $Z(t)$ is non-Gaussian, we operate so as to generate independent noise samples, z_i, $i = 1, N$ in time T, so that only first order pdf's are required. We now use the LOBD or threshold operation which we know becomes asymptotically optimum as our signal $S(t)$ becomes sufficiently small and $N \rightarrow \infty$ (Middleton and Spaulding, 1983). Increasing N corresponds to increasing the detection time T, since we cannot for any noise process sample more rapidly than the bandwidth and maintain independence. Using a vector Taylor expansion about the signals, $\underline{S_j}$, $j = 1, 2$ here, we get

$$\begin{aligned} p_Z(\underline{X} - \underline{S_j}) = p_Z(\underline{X}) &- \sum_{i=1}^{N} \frac{\partial p_Z(\underline{X})}{\partial x_i} S_{ji} \\ &+ \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} \frac{\partial^2 p_Z(\underline{X})}{\partial x_i \, \partial x_k} S_{ji} S_{jk} + \cdots \end{aligned} \tag{3}$$

In this expansion, for coherent signaling, all signal terms of degree two and higher are discarded. This is the normal "small signal assumption." In general, simply discarding higher order terms can lead to receiver structures which are not locally optimum, or in the limit of infinitely large sample sizes ($N \to \infty$), are not asymptotically optimum detection algorithms (AODA's). The proper algorithms require a correct bias (obtainable from proper treatment of the higher order terms)*. The problem is, that without the proper bias, the higher-order terms in the expansion of $\Lambda(\underline{X})$ can be discarded only when the sample size N is small. But N must be made large in order to obtain the required small probabilities of error for weak signals. This, of course, defeats the whole concept of a canonical and comparatively simple algorithm. One may as well use $\Lambda(\underline{X})$ itself, which is optimum for all signal levels. Sufficient conditions that the LOBD is an AODA as well as a LOBD ($N < \infty$) are given in Middleton and Spaulding (1983), Sec. A.3-3.

For binary symmetric CPSK, and for independent noise samples (3) leads to

$$\Lambda(\underline{X}) \sim \frac{1 - \sum_{i=1}^{N} \frac{d}{dx_i} \ln p_Z(x_i) S_{2i}}{1 - \sum_{i=1}^{N} \frac{d}{dx_i} \ln p_Z(x_i) S_{1i}} \quad \underset{H_2}{\overset{H_1}{\lessgtr}} \quad 1, \tag{4}$$

which gives the well-known receiver structure shown in Figure 1. In Figure 1, we see that the receiver is the standard memoryless Gaussian (i.e., degenerate matched filter) preceded by a particular nonlinearity given by

$$\ell(x) = - \frac{d}{dx} \ln p_Z(x) \ . \tag{5}$$

Note that this is a completely canonical result in that we have not yet specified (in the above derivation) what $p_Z(z)$ is or what the signals $S_1(t)$, $S_2(t)$ are except that they are completely known. Figure 1 is our receiver, which is adaptive in that it must change according to (5) for changing noise conditions. The receiver takes our received waveform samples x_i and uses them as shown to determine our decision variable δ. Now, in order to determine performance we need the pdf of δ. The pdf of δ is almost always impossible to obtain, however, unless we can invoke the Central Limit Theorem.

* For cases of threshold signal detectors that are neither locally optimum or asymptotically optimum detection algorithms, see Lu and Eisenstein (1981).

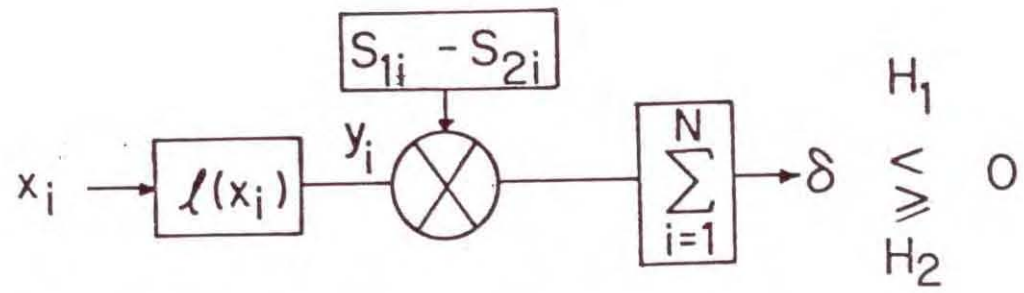

Figure 1. LOBD for binary symmetric purely coherent signals.

Although the nonlinearity $\ell(x)$ does not "Gaussianize" the noise, it does limit the amplitude excursions of the noise. Because of this, it is common to require N to be large (normally N must be relatively large to achieve any kind of processing gain over normal receivers as will be demonstrated later via simulation) so that we can apply the Central Limit Theorem. This means that we only need to compute the mean and variance of δ under each of the two hypotheses. We start with y_i, the output of the nonlinearity for input x_i. Suppose H_1 is true, then

$$E[y_i|H_1] = -\int_{-\infty}^{\infty} \frac{p_Z'(z)}{p_Z(z)}\, p_Z(z - S_{1i})\, dz.$$

and (6)

$$E[y_i^2|H_1] = \int_{-\infty}^{\infty} \left[\frac{p_Z'(z)}{p_Z(z)}\right]^2 p_Z(z - S_{1i})\, dz .$$

In evaluating the above two integrals, the usual approach is to expand the $p_Z(z - S_{1i})$ and then discard all terms in S_{1i} of degree 2 and higher. (As mentioned earlier, one of the objects is to investigate the effect of using this small signal assumption this second time.) Doing the above, we obtain

$$E[y_i|H_1] \doteq - S_{1i}L, \text{ where} \tag{7}$$

$$L = \int_{-\infty}^{\infty} \frac{[p_Z'(z)]^2}{p_Z(z)}\, dz, \text{ and} \tag{8}$$

$$E[y_i^2|H_1] \doteq L. \tag{9}$$

The parameter L determines (for "small"signal) the processing gain achievable for any $p_Z(z)$, including Gaussian noise (for which L = 1).

Using the above we obtain, for binary symmetric signal with $S_1(t) = -S_2(t)$ (CPSK),

$$E[\delta|H_2] = -E[\delta|H_1] = 2L \sum_{i=1}^{N} S_{1i}^2 \text{ , and}$$

$$\mathrm{Var}[\delta|H_2] = \mathrm{Var}[\delta|H_1] = 4 \sum_{i=1}^{N} (LS_{1i}^2 - L^2 S_{1i}^4). \tag{10}$$

An estimate of performance is then given by

$$P_e = \mathrm{Prob}[\delta < o] = \frac{1}{2} \operatorname{erfc} \left\{ \frac{|E[\delta]|}{\sqrt{2\mathrm{Var}[\delta]}} \right\} \quad . \tag{11}$$

If our two signals are, for example,

$$S_1(t) = \sqrt{2S} \cos(\omega_o t), \qquad 0 \leq t < T$$

and

$$S_2(t) = -\sqrt{2S} \cos(\omega_o t), \qquad 0 \leq t < T \tag{12}$$

so that S is the signal power, then

$$E[\delta] = 2SLN, \text{ and} \tag{13}$$

$$\mathrm{Var}[\delta] = 4SLN - 6S^2L^2N. \tag{14}$$

Since all our noise models are normalized so that the noise power = 1, S is also our signal-to-noise ratio. We note that SL must be such that the variance is positive. Since L is usually large (i.e., $\sim 10^3 - 10^4$), (14) defines, in a sense, the meaning of "small" signal in the above LOBD analysis. [Very detailed definitions of "small signal" are given in Middleton and Spaulding (1983), cf. Sections 7.4,A.3.] If SL << 1, then (11) becomes approximately

$$P_e \cong 1/2 \operatorname{erfc} (\sqrt{SNL/2}) \quad . \tag{15}$$

For LOBD's, the performance parameter L is ≥ 1, and is equal to 1 iff the noise is Gaussian.

The above reviews the LOBD approach. Suppose now that we have a LOBD detector based on the assumption that our interference is $\hat{p}_Z(z)$, and the actual interference is $p_Z(z)$. We can carry out the above analysis using $\hat{p}_Z(z)$ in place of $p_Z(z)$ where appropriate to determine the effects of "mismatching" the interference, or we can use this to determine the sensitivity of the LOBD performance to changing interference. This approach also gives results which can be easily used to evaluate the small signal performance of any ad hoc nonlinearity. The result is that L is replaced by a parameter L_{eff}, for "L effective," where, $L_{eff} = L_1^2/L_2$,

$$L_1 = \int_{-\infty}^{\infty} \left[\frac{\hat{p}_Z'(z)}{\hat{p}_Z(z)}\right] p_Z'(z)dz, \qquad \text{and} \tag{16}$$

$$L_2 = \int_{-\infty}^{\infty} \left[\frac{\hat{p}_Z'(z)}{\hat{p}_Z(z)}\right]^2 p_Z(z)dz \ . \tag{17}$$

If $\hat{p}_Z(z) = p_Z(z)$, then $L_1 = L_2 = L = L_{eff}$.

We can quickly compute the performance of any arbitrary nonlinearity, $\ell(x)$, used in the detector of Figure 1. For example, for the hard-limiter, $\ell(x) = 1$, if $x \geq 0$ and $\ell(x) = -1$, if $x < 0$. We can solve the resulting expression

$$\ell(x) = -\frac{d}{dx} \ell n\ \hat{p}_Z(x) \ , \tag{18}$$

to obtain the corresponding $\hat{p}_Z(z)$ to compute L_{eff} via (16) and (17) above. For the hard-limiter case, we obtain

$$L_{eff} = 4\ p_Z^2(0). \tag{19}$$

where $p_Z(z)$ is the actual interference. Performance is given by (15), so that the degradation caused by using the hard-limiter is simply the difference between L for our actual interference (LOBD performance factor) and L_{eff} for the hard-limiter (or similarly, for any other nonlinearity).

Up to this point, we have not specified any "model" for the real world non-Gaussian noise and interference environment. Recent work by Middleton has led to the development of a physical-statistical model for radio noise. This model has been used to develop optimum detection algorithms for a wide range of communications problems (Spaulding and Middleton , 1977). It is this model which we use here for our signal detection problem. The Middleton model is the only general one proposed to date in which the parameters of the model are determined explicitly by the underlying physical mechanisms (e.g., source density, beam-patterns, propagation conditions, emission waveforms, etc.). It is also the first model which treats narrowband interference processes (termed Class A), as well as the traditional broadband processes (Class B). The model is also canonical in nature in that the mathematical forms do not change with changing physical conditions. For a large number of comparisons of the model with measurements and for the details of the derivation of the model, see Middleton (1977,1983) and Spaulding (1977). We only summarize the results of the model which we need here.

For our received noise process Z(t), the probability density function (pdf) for the received instantaneous amplitude, z, is:

$$p_Z(z) = \frac{e^{-z^2/\Omega}}{\pi\sqrt{\Omega}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} A_\alpha^m \Gamma\left(\frac{m\alpha+1}{2}\right) {}_1F_1\left(-\frac{m\alpha}{2};\ 1/2;\ \frac{z^2}{\Omega}\right), \quad -\infty \le z \le \infty \tag{20}$$

where ${}_1F_1$ is a confluent hypergeometric function. The model has three parameters: α, A_α, and Ω. [A more detailed and complete model involving additional parameters has been developed, but (20) above is quite sufficient for our purposes]. The parameters α and A_α are intimately involved in the physical processes causing the interference. Again, definitions and details are contained in the references. The parameter Ω is a normalizing parameter. In the references, the normalization is $\Omega = 1$, which normalizes the process to the energy contained in the Gaussian portion of the noise. Here we use a value of Ω which normalizes the process (z values) to the measured energy in the process. We cannot normalize to the computed energy, since for (1), the second moment (or any moment) does not exist (i.e., is infinite). This is a typical problem with most such models for broadband impulsive noise. While the more complete model removes this problem, use of (20) in conjunction with measured data, will in no way limit us. However, when we discuss the simulation results, we will see an interesting result of using "infinite energy" models.

The result corresponding to (20) for the envelope cumulative distribution (APD) is:

$$P(E > E_o) = e^{-E_o^2/\Omega}\left[1 - \frac{E_o^2}{\Omega}\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} A_\alpha^m \right. \tag{21}$$

$$\left. \times\ \Gamma\left(1 + \frac{m\alpha}{2}\right) {}_1F_1\left(1 - \frac{m\alpha}{2};\ 2;\ \frac{E_o^2}{\Omega}\right)\right]$$

$$0 \leq E \leq \infty \ .$$

It is the envelope distribution in the above form which is usually measured and which we use for validation of the model by comparison with measurements.

The corresponding expressions for the Class A, narrowband "impulsive" noise are

$$P_Z(z) = e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-z^2/2\sigma_m^2} \quad , \tag{22}$$

where

$$\sigma_m^2 = \frac{m/A + \Gamma'}{1 + \Gamma'} \quad , \tag{23}$$

and, for the envelope,

$$P(E > E_o) = e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} e^{-E_o^2/\sigma_m^2} \quad . \tag{24}$$

The Class A model has two parameters: A and Γ'. A is termed the overlap index, and as A becomes large ($\sim$10), the noise approaches Gaussian (still narrowband) and Γ' is the ratio of the energy in the Gaussian portion of the noise to the energy in the non-Gaussian component.

The Class A model is appropriate for interference caused by collections of intentionally-radiated signals (e.g., as in the crowded HF band) and has also found application in various acoustical (e.g., sonar) problems. The Class B model is appropriate for broadband impulsive noise processes such as atmospheric noise, automotive ignition noise, etc.

Figures 2 and 3 show the comparison of the limiting small signal performance for the LOBD with the corresponding performance for the hard limiter. Figure 2 is for the Class B Middleton model for a wide range of the parameters α and A_α, and Figure 3 is for the Class A model for various values of the parameters A and Γ'. A couple of example values for L are also shown on the figures. On Figure 2, the point shown ($\alpha = 1$, $A_\alpha = 1$) will be used and referred to later.

While the hard-limiter may not be the suboptimum nonlinearity one would choose for all Class A cases, the results show that the Class A LOBD nonlinearity can substantially outperform the hard-limiter (Figure 3). The results for Class B noise would seem to indicate that one may as well use a hard-limiter rather than attempting to implement the much more difficult Class B LOBD nonlinearity. The results, however, are limiting results for a suitably small signal (>0) and $N \to \infty$.

Note that if we use the receiver optimum for Gaussian noise (no nonlinearity), the limiting performance ($N \to \infty$) is identical for all types of noise (i.e., $L_{eff} = 1$ from (16) and (17) for <u>any</u> $p_Z(z)$). While this is certainly true in the limit, we also know that performance of systems, using the Gaussian receiver in non-Gaussian noise, can be quite different, even for very small signals. This means that, in this case at least, the limiting performance may not give a good estimate for real-world small signal situations, especially for relatively small N. For large N, and any noise process, we expect the performance to approach the same characteristic performance as for Gaussian noise due to the Central Limit Theorem. However, we have no means of locating this "Gaussian performance curve" in terms of signal-to-noise ratio. [For our CPSK case, we will see that the parameter L is a measure of the difference (in the limit as $N \to \infty$) between the LOBD "Gaussian performance curve" and the linear receiver "Gaussian performance curve," where both are operating in the same non-Gaussian noise environment.] Figure 2 shows that the Class B LOBD nonlinearity and the hard-limiter nonlinearity behave similarly (only small degradation); however, these results may be true only in the limit. In the next section, we investigate this question.

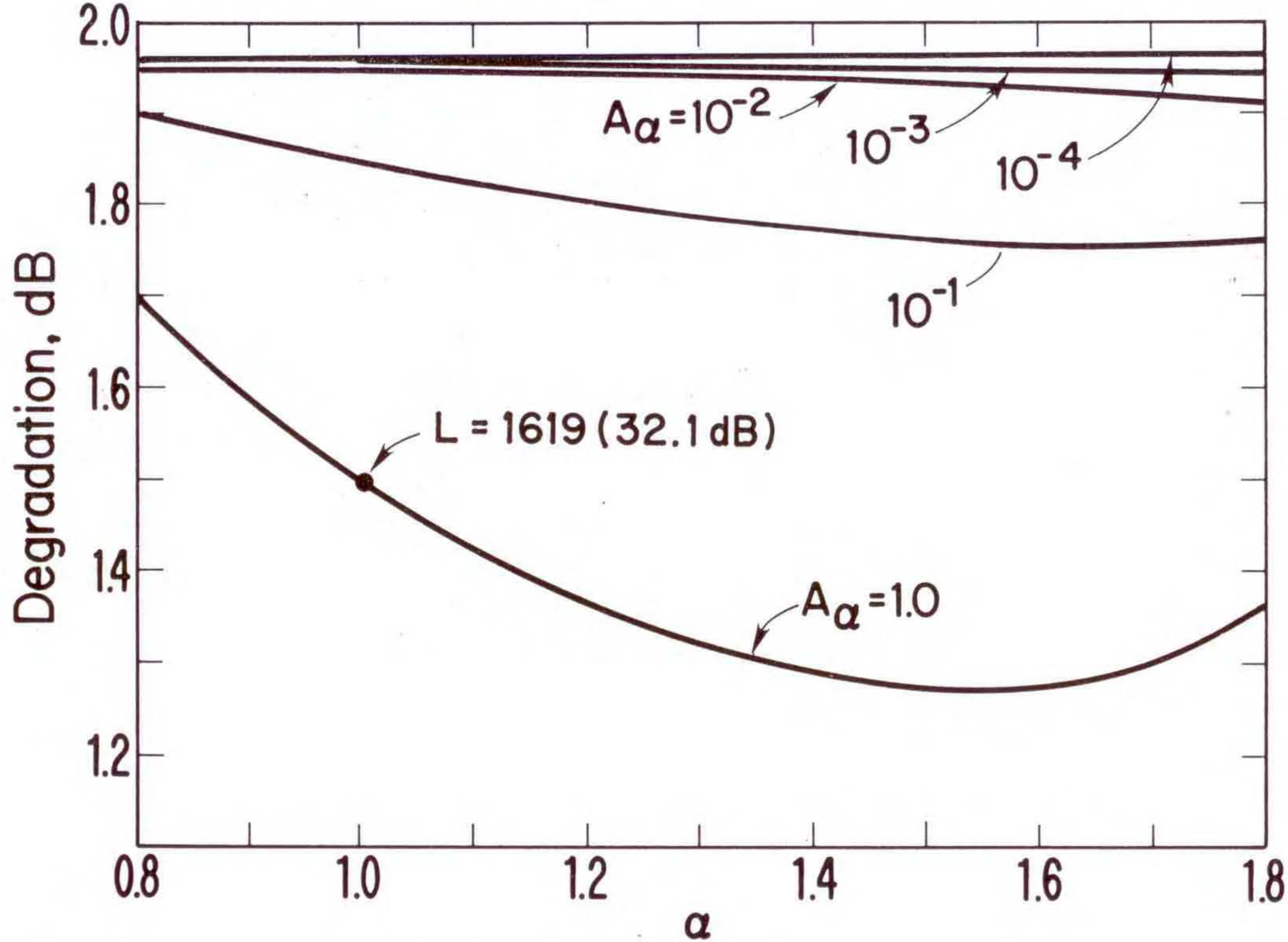

Figure 2. Comparison of the optimum nonlinearity for Class B noise with the hard-limiter.

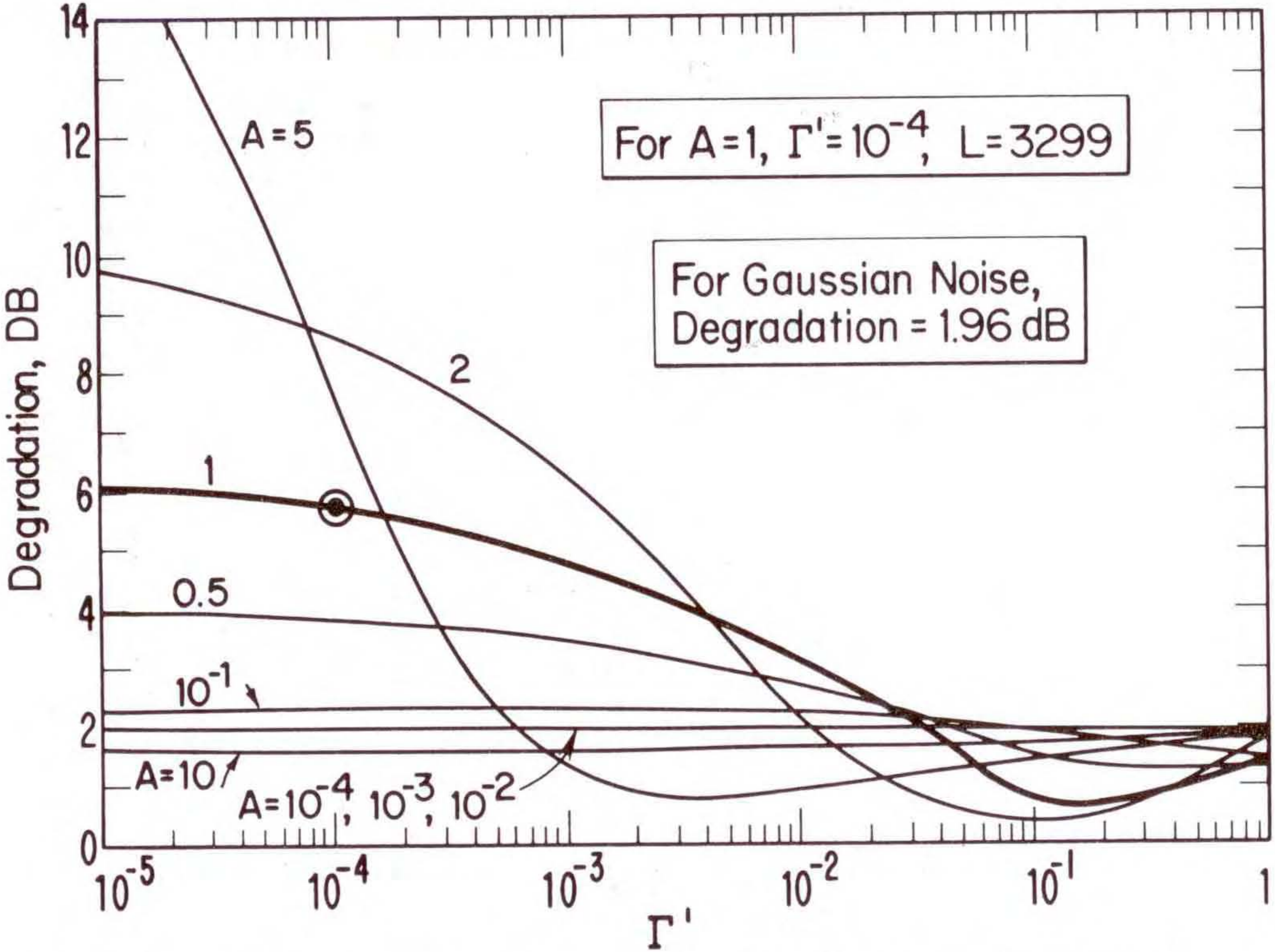

Figure 3. Comparison of the optimum nonlinearity for Class A noise with the hard-limiter.