APPENDIX A

INTERPOLATION IN A TRIANGLE

Assuming that the plane is divided into a number of triangles, we describe a procedure for interpolating values of a function in each triangle. The primary emphasis is on the smoothness of the interpolated values not only inside of the triangle but also on the side of it; i.e., the interpolated values in a triangle must smoothly connect with those values in an adjacent triangle on the common side of two triangles.

Basic Assumptions.

Using a two-dimensional Cartesian coordinate system with x and y axes, we describe the basic assumptions as follows:

(i) The value of the function at point (x, y) in a triangle is interpolated by a bivariate fifth-degree polynomial in x and y; i.e.,

$$z(x, y) = \sum_{j=0}^{5} \sum_{k=0}^{5-j} q_{jk} x^j y^k . \qquad \text{(A-1)}$$

Note that there are 21 coefficients to be determined.

(ii) The values of the function and its first-order and second-order partial derivatives (i.e., z, z_x, z_y, z_{xx}, z_{xy}, and z_{yy}) are given at each vertex of the triangle. This assumption yields 18 independent conditions.

(iii) The partial derivative of the function differentiated in the direction perpendicular to each side of the triangle is a polynomial of degree three, at most, in the variable measured in the direction of the side of the triangle. In other words, when the coordinate system is transformed to another Cartesian system, which we call the s-t system, in such a way that the s axis is parallel to each of the side of the triangle, the bivariate polynomial in s and t representing the z values must satisfy

$$z_{tssss} = 0. \tag{A-2}$$

Since a triangle has three sides, this assumption yields three additional conditions.

The purpose of the third assumption is two-fold. This assumption adds three independent conditions to the 18 conditions dictated by the second assumption and, thus, enables one to determine the 21 coefficients of the polynomial. It also assures smoothness of interpolated values as described in the following paragraph.

We will prove smoothness of the interpolated values and therefore smoothness of the resulting surface along the side of the triangle. Since the coordinate transformation between the x-y system and the s-t system is linear, the values of z_x, z_y, z_{xx}, z_{xy}, and z_{yy} at each vertex uniquely determine the values of z_s, z_t, z_{ss}, z_{st}, and z_{tt} at the same vertex, each of the latter as a linear combination of the former. Then, the z, z_s, and z_{ss} values at two vertexes uniquely determine a fifth-degree polynomial in s for z on the side between these vertexes. Since two fifth degree polynomials in x and y representing z values in two triangles that share the common side are reduced to fifth-degree polynomials in s on the side, these two polynomials in x and y coincide with each other on the common side. This proves continuity of the interpolated z values along a side of a triangle. Similarly, the values of z_t and $z_{st} = (z_t)_s$ at two vertexes uniquely determine a third-degree polynomial in s for z_t on the side. Since the polynomial representing z_t is assumed to be third degree at most with respect to s, two polynomials representing z_t in two triangles that share the common side also coincide with each other on the side. This proves continuity of z_t and thus smoothness of z along the side of the triangle.

Coordinate System Associated With the Triangle.

We denote the vertexes of the triangle by V_1, V_2, and V_3 in a counter-clockwise order, and their respective coordinates in the x-y Cartesian coordinate system by (x_1, y_1), (x_2, y_2), and (x_3, y_3), as shown in figure A-1(a). We introduce a new coordinate system associated with the triangle, where the vertexes are represented by (0, 0), (1, 0), and (0, 1) as shown in figure A-1(b). We call this new system the u-v system.

The coordinate transformation between the x-y system and the u-v system is represented by

$$\begin{aligned} x &= au + bv + x_0 , \\ y &= cu + dv + y_0 , \end{aligned} \tag{A-3}$$

where

$$\begin{aligned} a &= x_2 - x_1 , \\ b &= x_3 - x_1 , \\ c &= y_2 - y_1 , \\ d &= y_3 - y_1 , \\ x_0 &= x_1 , \\ y_0 &= y_1 . \end{aligned} \tag{A-4}$$

The inverse relation is

$$\begin{aligned} u &= [\ d(x - x_0) - b(y - y_0)] / (ad - bc) , \\ v &= [-c(x - x_0) + a(y - y_0)] / (ad - bc) . \end{aligned} \tag{A-5}$$

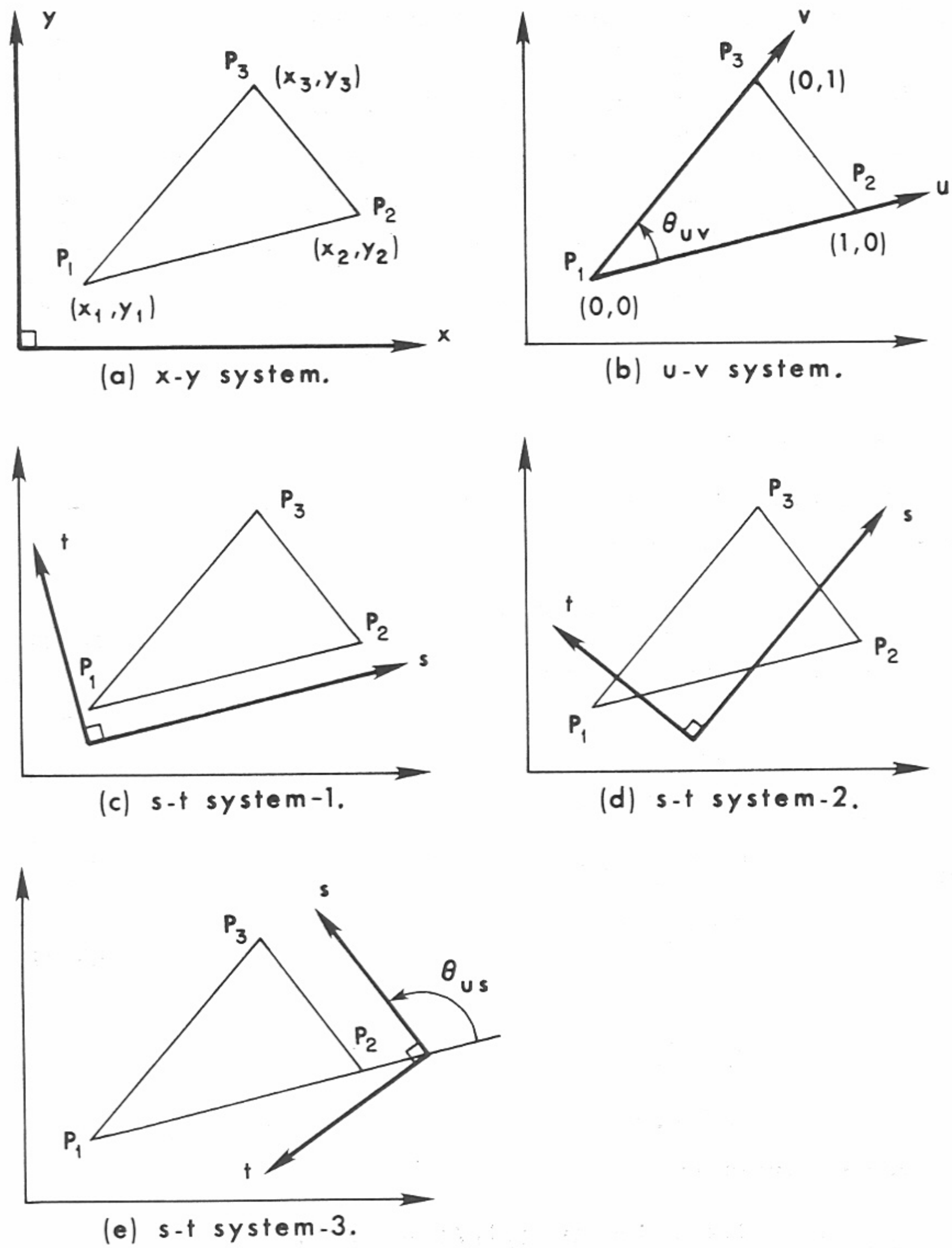

Figure A-1. Various coordinate systems.

The partial derivatives in the x-y system are transformed to the u-v system by

$$
\begin{aligned}
z_u &= a\,z_x + c\,z_y\,, \\
z_v &= b\,z_x + d\,z_y\,, \\
z_{uu} &= a^2\,z_{xx} + 2\,a\,c\,z_{xy} + c^2\,z_{yy}\,, \\
z_{uv} &= a\,b\,z_{xx} + (a\,d + b\,c)\,z_{xy} + c\,d\,z_{yy}\,, \\
z_{vv} &= b^2\,z_{xx} + 2\,b\,d\,z_{xy} + d^2\,z_{yy}\,.
\end{aligned}
\tag{A-6}
$$

Since this coordinate transformation is linear, the interpolating polynomial (A-1) is transformed to

$$
z(u,v) = \sum_{j=0}^{5} \sum_{k=0}^{5-j} p_{jk}\,u^j\,v^k\,. \tag{A-7}
$$

Since it is the p coefficients that are determined directly, as shown later, and are used for interpolating z values, it is unnecessary to relate the p coefficients to the q coefficients used in (A-1).

The partial derivatives of $z(u,v)$ in the u-v system are expressed by

$$
\begin{aligned}
z_u(u,v) &= \sum_{j=1}^{5} \sum_{k=0}^{5-j} j\,p_{jk}\,u^{j-1}\,v^k\,, \\
z_v(u,v) &= \sum_{j=0}^{4} \sum_{k=1}^{5-j} k\,p_{jk}\,u^j\,v^{k-1}\,, \\
z_{uu}(u,v) &= \sum_{j=2}^{5} \sum_{k=0}^{5-j} j\,(j-1)\,p_{jk}\,u^{j-2}\,v^k\,, \\
z_{uv}(u,v) &= \sum_{j=1}^{4} \sum_{k=1}^{5-j} j\,k\,p_{jk}\,u^{j-1}\,v^{k-1}\,,
\end{aligned}
\tag{A-8}
$$

$$z_{vv}(u,v) = \sum_{j=0}^{3} \sum_{k=2}^{5-j} k(k-1)\, p_{jk}\, u^j v^{k-2} .$$

We denote the lengths of the unit vectors in the u-v system (i.e., the lengths of sides $\overline{V_1 V_2}$ and $\overline{V_1 V_3}$) by L_u and L_v, respectively, and the angle between the u and v axes by θ_{uv}. They are given by

$$\begin{aligned} L_u &= a^2 + c^2 , \\ L_v &= b^2 + d^2 , \\ \theta_{uv} &= \tan^{-1}(d/b) - \tan^{-1}(c/a) , \end{aligned} \qquad (A\text{-}9)$$

where a, b, c, and d are constants given in (A-4).

Implementation of the Third Assumption.

We represent the third assumption (A-2) in the u-v system and derive useful equations for determining the coefficients of the polynomial. We do this for three cases corresponding to the three sides of the triangle.

First, we consider the case where the s axis is parallel to side $\overline{V_1 V_2}$, as shown in figure A-1(c). The coordinate transformation between the u-v system and the s-t system is expressed by

$$\begin{aligned} u &= [(\sin\theta_{uv})(s - s_0) - (\cos\theta_{uv})(t - t_0)] / (L_u \sin\theta_{uv}) , \\ v &= (t - t_0) / (L_v \sin\theta_{uv}) , \end{aligned} \qquad (A\text{-}10)$$

where L_u, L_v, and θ_{uv} are constants given in (A-9). Partial derivatives with respect to s and t are expressed by

$$\begin{aligned} \frac{\partial}{\partial s} &= \frac{1}{L_u} \frac{\partial}{\partial u} , \\ \frac{\partial}{\partial t} &= - \frac{\cos\theta_{uv}}{L_u \sin\theta_{uv}} \frac{\partial}{\partial u} + \frac{1}{L_v \sin\theta_{uv}} \frac{\partial}{\partial v} , \end{aligned} \qquad (A\text{-}11)$$

respectively. From (A-2), (A-7), and (A-11), we obtain

$$L_u p_{41} - 5 L_v \cos\theta_{uv} p_{50} = 0 . \tag{A-12}$$

Next, we consider the case where the s axis is parallel to side $\overline{V_1 V_3}$, as shown in figure A-1(d). The coordinate transformation is expressed by

$$\begin{aligned} u &= -(t - t_0) / (L_u \sin\theta_{uv}) , \\ v &= [(\sin\theta_{uv})(s - s_0) + (\cos\theta_{uv})(t - t_0)] / (L_v \sin\theta_{uv}) . \end{aligned} \tag{A-13}$$

Partial derivatives are expressed by

$$\begin{aligned} \frac{\partial}{\partial s} &= \frac{1}{L_v} \frac{\partial}{\partial v} , \\ \frac{\partial}{\partial t} &= -\frac{1}{L_u \sin\theta_{uv}} \frac{\partial}{\partial u} + \frac{\cos\theta_{uv}}{L_v \sin\theta_{uv}} \frac{\partial}{\partial v} . \end{aligned} \tag{A-14}$$

Then, from (A-2), (A-7), and (A-14), we obtain

$$L_v p_{14} - 5 L_u \cos\theta_{uv} p_{05} = 0 . \tag{A-15}$$

Next, we consider the third case where the s axis is parallel to side $\overline{V_2 V_3}$, as shown in figure A-1(e). The coordinate transformation is expressed by

$$\begin{aligned} u &= A(s - s_0) + B(t - t_0) , \\ v &= C(s - s_0) + D(t - t_0) , \end{aligned} \tag{A-16}$$

where

$$\begin{aligned} A &= \sin(\theta_{uv} - \theta_{us}) / (L_u \sin\theta_{uv}) , \\ B &= -\cos(\theta_{uv} - \theta_{us}) / (L_u \sin\theta_{uv}) , \\ C &= \sin\theta_{us} / (L_v \sin\theta_{uv}) , \\ D &= \cos\theta_{us} / (L_v \sin\theta_{uv}) , \end{aligned} \tag{A-17}$$

$$\theta_{us} = \tan^{-1}[(d-c)/(b-a)] - \tan^{-1}(c/a).$$

The θ_{us} constant is the angle between the s and the u axes. The a, b, c, and d constants are given in (A-4), and L_u, L_v, and θ_{uv} are given in (A-9). Partial derivatives with respect to s and t are expressed by

$$\frac{\partial}{\partial s} = A\frac{\partial}{\partial u} + C\frac{\partial}{\partial v},$$
$$\frac{\partial}{\partial t} = B\frac{\partial}{\partial u} + D\frac{\partial}{\partial v}. \qquad \text{(A-18)}$$

From (A-2), (A-7), and (A-18), we obtain

$$5A^4Bp_{50} + A^3(4BC+AD)p_{41} + A^2C(3BC+2AD)p_{32}$$
$$+ AC^2(2BC+3AD)p_{23} + C^3(BC+4AD)p_{14} + 5C^4Dp_{05} = 0. \qquad \text{(A-19)}$$

Equations (A-12), (A-15), and (A-19) are the results of implementation of the third assumption (A-2) in the u-v coordinate system. They are used for determining the coefficients of the polynomial (A-7).

Determination of the Coefficients of the Polynomial.

Obviously, we can determine the coefficients of the lower-power terms by letting u = 0 and v= 0 and by inserting the values of z, z_u, z_v, z_{uu}, z_{uv}, and z_{vv} at V_1 (i.e., u = 0 and v = 0) in (A-7) and (A-8). The results are

$$p_{00} = z(0,0),$$
$$p_{10} = z_u(0,0),$$
$$p_{01} = z_v(0,0), \qquad \text{(A-20)}$$
$$p_{20} = z_{uu}(0,0)/2,$$

$$p_{11} = z_{uv}(0,0) ,$$

$$p_{02} = z_{vv}(0,0)/2 .$$

Next, letting u = 1 and v = 0 and inserting the values of z, z_u, and z_{uu} at V_2 (i.e., u = 1 and v = 0) in (A-7) and the first and the third equations in (A-8), we obtain the following three equations:

$$\begin{aligned} p_{30} + \quad p_{40} + \quad p_{50} &= z(1,0) - p_{00} - p_{10} - p_{20} , \\ 3\,p_{30} + \ 4\,p_{40} + \ 5\,p_{50} &= z_u(1,0) - p_{10} - 2\,p_{20} , \\ 6\,p_{30} + 12\,p_{40} + 20\,p_{50} &= z_{uu}(1,0) - 2\,p_{20} . \end{aligned}$$

Solving these equations with respect to p_{30}, p_{40}, and p_{50}, we obtain

$$\begin{aligned} p_{30} &= [20\,z(1,0) - 8\,z_u(1,0) + z_{uu}(1,0) - 20\,p_{00} - 12\,p_{10} - 6\,p_{20}]/2, \\ p_{40} &= -15\,z(1,0) + 7\,z_u(1,0) - z_{uu}(1,0) + 15\,p_{00} + 8\,p_{10} + 3\,p_{20} , \\ p_{50} &= [12\,z(1,0) - 6\,z_u(1,0) + z_{uu}(1,0) - 12\,p_{00} - 6\,p_{10} - 2\,p_{20}]/2 . \end{aligned} \tag{A-21}$$

Since p_{00}, p_{10}, and p_{20} are already determined by (A-20), we can calculate p_{30}, p_{40}, and p_{50} from (A-21).

Similarly, using the values of z, z_v, and z_{vv} at V_3 (i.e., u = 0 and v = 1) and working with (A-7) and the second and the last equations in (A-8), we obtain

$$\begin{aligned} p_{03} &= [20\,z(0,1) - 8\,z_v(0,1) + z_{vv}(0,1) - 20\,p_{00} - 12\,p_{01} - 6\,p_{02}]/2, \\ p_{04} &= -15\,z(0,1) + 7\,z_v(0,1) - z_{vv}(0,1) + 15\,p_{00} + 8\,p_{01} + 3\,p_{02} , \\ p_{05} &= [12\,z(0,1) - 6\,z_v(0,1) + z_{vv}(0,1) - 12\,p_{00} - 6\,p_{01} - 2\,p_{02}]/2 . \end{aligned} \tag{A-22}$$

With p_{50} and p_{05} determined, we can determine p_{41} and p_{14} from (A-12) and (A-15), respectively. The results are

$$p_{41} = \frac{5 L_v \cos\theta_{uv}}{L_u} p_{50} ,$$

$$p_{14} = \frac{5 L_u \cos\theta_{uv}}{L_v} p_{05} . \tag{A-23}$$

Next, we use the values of z_v and z_{uv} at V_2 (i.e., u = 1 and v = 0) with the second and the fourth equations in (A-8) and obtain

$$p_{21} + p_{31} = z_v(1,0) - p_{01} - p_{11} - p_{41} ,$$

$$2 p_{21} + 3 p_{31} = z_{uv}(1,0) - p_{11} - 4 p_{41} .$$

Solving these equations, we obtain

$$\begin{aligned} p_{21} &= 3 z_v(1,0) - z_{uv}(1,0) - 3 p_{01} - 2 p_{11} + p_{41} , \\ p_{31} &= - 2 z_v(1,0) + z_{uv}(1,0) + 2 p_{01} + p_{11} - 2 p_{41} . \end{aligned} \tag{A-24}$$

Similarly, using the values of z_u and z_{uv} at V_3 (i.e., u = 0 and v = 1) with the first and the fourth equations in (A-8), we obtain

$$\begin{aligned} p_{12} &= 3 z_u(0,1) - z_{uv}(0,1) - 3 p_{10} - 2 p_{11} + p_{14} , \\ p_{13} &= - 2 z_u(0,1) + z_{uv}(0,1) + 2 p_{10} + p_{11} - 2 p_{14} . \end{aligned} \tag{A-25}$$

Equation (A-19) is rewritten as

$$g_1 p_{32} + g_2 p_{23} = h_1 , \tag{A-26}$$

where

$$g_1 = A^2 C (3 B C + 2 A D) ,$$

$$g_2 = A C^2 (2 B C + 3 A D) , \tag{A-27}$$

$$h_1 = -5A^4 B p_{50} - A^3(4BC + AD)p_{41}$$
$$- C^3(BC + 4AD)p_{14} - 5C^4 D p_{05} ,$$

with A, B, C, and D defined by (A-17). From the value of z_{vv} at V_2 and the last equation in (A-8), we obtain

$$p_{22} + p_{32} = h_2 , \tag{A-28}$$

where

$$h_2 = (1/2)\, z_{vv}(1,0) - p_{02} - p_{12} . \tag{A-29}$$

Similarly, from the value of z_{uu} at V_3 and the third equation in (A-8), we obtain

$$p_{22} + p_{23} = h_3 , \tag{A-30}$$

where

$$h_3 = (1/2)\, z_{uu}(0,1) - p_{20} - p_{21} . \tag{A-31}$$

Solving (A-26), (A-28), and (A-30) with respect to p_{22}, p_{32}, and p_{23}, we obtain

$$\begin{aligned} p_{22} &= (g_1 h_2 + g_2 h_3 - h_1)/(g_1 + g_2) , \\ p_{32} &= h_2 - p_{22} , \\ p_{23} &= h_3 - p_{22} , \end{aligned} \tag{A-32}$$

with g_1, g_2, h_1, h_2, and h_3 given by (A-27), (A-29), and (A-31).

Step-by-Step Description of the Procedure.

In summary, the coefficients of the polynomial are determined by the following steps:

(i) Determine a, b, c, and d (coefficients for coordinate transformation) from (A-4).

(ii) Calculate partial derivatives z_u, z_v, z_{uu}, z_{uv}, and z_{vv} from (A-6).

(iii) Calculate L_u, L_v, and θ_{uv} (constants associated with the u-v coordinate system) from (A-9).

(iv) Calculate A, B, C, and D (coefficients for another coordinate transformation) from (A-17).

(v) Determine 18 coefficients of the polynomial from (A-20), (A-21), (A-22), (A-23), (A-24), and (A-25) -- in this order.

(vi) Calculate g_1, g_2, h_1, h_2, and h_3 from (A-27), (A-29), and (A-31).

(vii) Determine the remaining three coefficients from (A-32).

For a given point (x, y) in the triangle, one can interpolate the z value by the following steps:

(i) Transform x and y to u and v by (A-5) with necessary coefficients given by (A-4).

(ii) Evaluate the polynomial for z(u, v) given in (A-7).

Although some equations look complicated, the procedure described here is straightforward. It can easily be implemented as a computer subroutine.