OT REPORT 75-70

A METHOD OF BIVARIATE INTERPOLATION AND SMOOTH SURFACE FITTING FOR VALUES GIVEN AT IRREGULARLY DISTRIBUTED POINTS

HIROSHI AKIMA

U.S. DEPARTMENT OF COMMERCE
Rogers C. B. Morton, Secretary

Betsy Ancker-Johnson, Ph. D.
Assistant Secretary for Science and Technology

OFFICE OF TELECOMMUNICATIONS
John M. Richardson, Acting Director

August 1975

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402

UNITED STATES DEPARTMENT OF COMMERCE
OFFICE OF TELECOMMUNICATIONS
STATEMENT OF MISSION

The mission of the Office of Telecommunications in the Department of Commerce is to assist the Department in fostering, serving, and promoting the nation's economic development and technological advancement by improving man's comprehension of telecommunication science and by assuring effective use and growth of the nation's telecommunication resources.

In carrying out this mission, the Office

- Conducts research needed in the evaluation and development of policy as required by the Department of Commerce
- Assists other government agencies in the use of telecommunications
- Conducts research, engineering, and analysis in the general field of telecommunication science to meet government needs
- Acquires, analyzes, synthesizes, and disseminates information for the efficient use of the nation's telecommunication resources.
- Performs analysis, engineering, and related administrative functions responsive to the needs of the Director of the Office of Telecommunications Policy, Executive Office of the President, in the performance of his responsibilities for the management of the radio spectrum
- Conducts research needed in the evaluation and development of telecommunication policy as required by the Office of Telecommunications Policy, pursuant to Executive Order 11556

USCOMM - ERL

TABLE OF CONTENTS

A METHOD OF BIVARIATE INTERPOLATION AND SMOOTH SURFACE FITTING FOR VALUES GIVEN AT IRREGULARLY DISTRIBUTED POINTS

Hiroshi Akima *

Abstract — A method of bivariate interpolation and smooth surface fitting is developed for z values given at points irregularly distributed in the x-y plane. The interpolating function is a fifth-degree polynomial in x and y defined in each triangular cell which has projections of three data points in the x-y plane as its vertexes. Each polynomial is determined by the given values of z and estimated values of partial derivatives at the vertexes of the triangle. Procedures for dividing the x-y plane into a number of triangles, for estimating partial derivatives at each data point, and for determining the polynomial in each triangle are described. A simple example of the application of the proposed method is shown. User information and Fortran listings are given on a computer subprogram package that implements the proposed method.

Key Words and Phrases — Bivariate interpolation, interpolation, partial derivative, polynomial, smooth surface fitting.

---

* The author is with the Institute for Telecommunication Sciences, Office of Telecommunications, U.S. Department of Commerce, Boulder, Colorado 80302.

1. INTRODUCTION

In a previous study (Akima, 1974a,b), we developed a method of bivariate interpolation and smooth surface fitting. The method was designed in such a way that the resulting surface would pass through all the given data points. Adopting local procedures, it successfully suppressed undulations in the resulting surface which are very likely to appear in surfaces fitted by other methods. Like many other methods, however, this method also has a serious drawback. Applicability is restricted to cases where the values of the function are given at rectangular grid points in a plane; i.e., the values of $z = z(x, y)$ must be given as $z_{ij} = z(x_i, y_j)$ in the x-y plane, where $i = 1, 2, \ldots, n_x$ and $j = 1, 2, \ldots, n_y$. This restriction prevents application to cases where collection of data at rectangular grid points is impossible or otherwise impractical.

The subject of the present study is bivariate interpolation and smooth surface fitting in the general case where the values of the function are given at irregularly distributed points in a plane; i.e., the case where the z values are given as $z_i = z(x_i, y_i)$, where $i = 1, 2, \ldots, n$. Despite potentially wide applicability of a method of bivariate interpolation and smooth surface fitting for irregularly distributed points, studies for developing such a method have not been active in the past.

Two types of approaches are possible; one using a single global function, and the other based on a collection of local functions. In the former approach, the procedure often becomes too complicated to manage as the number of given data points increases. Moreover, the resulting surface from the former sometimes exhibits excessive undulations. For these reasons, only the latter approach is considered in the present study.

Bengtsson and Nordbeck (1964) suggested a method based on partitioning the x-y plane into a number of triangles (each triangle having projections of three data points in the x-y plane as its vertexes) and on fitting a plane to the surface in each triangle. Obviously, the resulting surface is not smooth on the sides of the triangles although it is continuous. In addition, their suggestion for partitioning so that the sum of the lengths of the sides of these triangles be minimized is too complicated to implement.

Shepard (1968) suggested a method based on weighted averages of the given z values. The basic weighting function is the square of the reciprocal of the distance between the projection of each data point and that of the point at which interpolation is to be performed. The actual weighting function is an improvement of this basic weighting function in that the actual function corresponding to a distant data point vanishes. Through this improvement the originally global procedures in this method became local. This method has several desirable properties. It takes into account the "shadowing" of the influence of a data point by a nearer one in the same direction. It yields reasonable slopes at the given data points. However, it fails to produce a plane when all the given data points lie in a slanted plane; this property is considered to be a serious drawback.

In conjunction with variational problems containing second-order derivatives, Zlamal (1968) discussed an approximation procedure using fifth-degree polynomials in x and y over triangular regions in the x-y plane. To determine the coefficients of the polynomial for each triangle, he uses, in addition to the z values and the first and second partial derivatives (i.e., z_x, z_y, z_{xx}, z_{xy}, and z_{yy}) at the three vertexes of the triangle, three partial derivatives, each differentiated in the direction normal to one of the three sides of the triangle at the midpoint of the side in question. The theory was generalized to $(4m+1)$st-degree polynomials for functions m-times continuously differentiable on a closed triangular domain by Zenisek (1970). Although a comprehensive interpolation method is not suggested in their papers, their papers were instrumental in stimulating portions of the ideas developed here.

In the present study, we develop and propose a method of bivariate interpolation and smooth surface fitting that is applicable to z values given at irregularly distributed points in the x-y plane. As in the method for rectangular grid points developed in the previous study (Akima, 1974 a, b), the interpolating function used in the method proposed in the present study is also a smooth function; i.e., the interpolating function and its first-order partial derivatives are continuous. The proposed method is also based on local procedures. The surface resulting from the proposed method will pass through all the given data points.

In this report, the proposed method is outlined in section 2, with some mathematical details in Appendix A. A simple example that illustrates the application of the proposed method is shown in section 3. Some pertinent remarks are addressed in section 4. In Appendix B, user information and Fortran listings are given on the IDBVIP/IDSFFT subprogram package that implements the proposed method.

2. DESCRIPTION OF THE METHOD

In this method the x-y plane is divided into a number of triangular cells; each having projections of three data points in the plane as its vertexes, and a bivariate fifth-degree polynomial in x and y is applied to each triangular cell.

For a unique partitioning of the plane, the x-y plane is divided into triangles by the following steps. First, determine the nearest pair of data points and draw a line segment between the points. Next, find the nearest pair of data points among the remaining pairs and draw a line segment between these points if the line segment to be drawn does not cross any other line segment already drawn. Repeat the second step until all possible pairs are exhausted.

The z value in a triangle is interpolated with a bivariate fifth-degree polynomial in x and y, i.e.,

$$z(x, y) = \sum_{j=0}^{5} \sum_{k=0}^{5-j} q_{jk} x^j y^k . \tag{1}$$

The coefficients of the polynomial are determined by the given z values at the three vertexes of the triangle and the estimated values of partial derivatives z_x, z_y, z_{xx}, z_{xy}, and z_{yy} at the vertexes, together with the imposed condition that the partial derivative of z by the variable measured in the direction perpendicular to each side of the triangle be a polynomial of degree three, at most, in the variable measured along the side. The procedure for interpolation in a triangle including determination of the coefficients of the polynomial is described in detail in Appendix A. Smoothness of the interpolated values and therefore smoothness of the resulting surface along each side of the triangle is proved also in the Appendix.

Procedures for estimating the five partial derivatives locally at each data point are not unique. The derivatives could be determined as partial derivatives of a second-degree polynomial in x and y that coincides with the given z values at six data points consisting of five data points the projections of which are nearest to the projection of the data point in question and the data point itself. This procedure is a bivariate extension of the one used in the univariate osculatory interpolation (Ackland, 1915). Adoption of this procedure has an advantage that, when z is a second-degree polynomial in x and y, the method yields exact results. As will be shown in section 3, however, this procedure sometimes yields very unreasonable results.

We will take a different approach and estimate the partial derivatives in two steps; i.e., the first-order derivatives in the first step and the second-order derivatives in the second step. To estimate the first-order partial derivatives at data point P_0 we use several additional data points P_i ($i = 1, 2, \ldots, n_n$) the projections of which are nearest to the projection of P_0 selected from all data points other than P_0. We take two data points P_i and P_j out of the n_n points and construct the vector product of $\overline{P_0P_i}$ and $\overline{P_0P_j}$; i.e., a vector that is perpendicular to both $\overline{P_0P_i}$ and $\overline{P_0P_j}$ with the right-hand rule and has a magnitude equal to the area of the parallelogram formed by $\overline{P_0P_i}$ and $\overline{P_0P_j}$. We take P_i and P_j in such a way that the resulting vector product always points upward (i.e., the z component of the vector product is always positive). We construct vector products for all possible combinations of $\overline{P_0P_i}$ and $\overline{P_0P_j}$ ($i \neq j$) and take a vector sum of all the vector products thus constructed. Then, we assume that the first-order partial derivatives z_x and z_y at P_0 are estimated as those of a plane that is normal to the resultant vector sum thus composed. Note that, when $n_n = 2$, the estimated z_x and z_y are equal to the partial derivatives of a plane that passes through P_0, P_1, and P_2. Also note that, when $n_n = 3$ and the projection of P_0 in the x-y plane lies inside the triangle formed by the projections of P_1, P_2, and P_3, the estimated z_x and z_y are equal to the partial derivatives of a plane that passes through P_1, P_2, and P_3.

In the second step, we apply the procedure of "partial differentiation" described in the preceding paragraph to the estimated z_x values at P_i ($i = 0, 1, 2, \ldots, n_n$) and obtain estimates of $z_{xx} = (z_x)_x$ and $z_{xy} = (z_x)_y$ at P_0. We repeat the same procedure for the estimated z_y values and obtain estimates of $z_{xy} = (z_y)_x$ and $z_{yy} = (z_y)_y$. We adopt a simple arithmetic mean of two z_{xy} values thus estimated as our estimate for z_{xy} at P_0.

The selection of n_n is again not unique. Obviously, n_n cannot be less than 2. Also, it must be less than the total number of data points. Other than those, there seems to exist no theory that dictates a definite value for n_n. The best we can say is that, based on the example to be shown in section 3 and on some others, we recommend a number between 3 and 5 (inclusive) for n_n.

3. APPLICATIONS

Using a simple example taken from the previous study (Akima, 1974 a, b), we illustrate the application of the proposed method. We take a quarter of the surface shown in the example in the previous study and sample 50 data points from the surface randomly. The coordinate values of the sampled data points are shown in table 1. Knowing from the physical nature of the phenomenon that z(x, y) is a single-valued smooth function of x and y, we try to interpolate the z values and to fit a smooth surface to the given data points.

Figure 1 depicts contour maps of the surfaces resulting from the 30 data points with asterisks in table 1, while figure 2, from all the 50 data points in the table. In these contour maps, projections of the data points are marked with encircled points. In each figure, the original surface from which the data points were sampled is shown in (a). The surface fitted with piecewise planes (i.e., the surface consisting of a number of pieces of planes, each applicable to one triangle) is shown in (b). Of course, such a surface is continuous but not smooth. The surface fitted by the method that estimates the partial derivatives with a second-degree polynomial is shown in (c). The surfaces fitted by the proposed method using three, four, and five additional data points for estimation of partial derivatives at each data point are shown in (d), (e), and (f), respectively. In drawing these contour maps, the z values were interpolated by their respective methods at the nodes of a grid consisting of 100 by 80 squares; in each square, the z values were interpolated linearly.

Figures 1 and 2 indicate that the proposed method yields reasonable results although these results might not necessarily be satisfactory for some applications. In these figures very little difference is

Table 1. An example set of data points.

(Thirty points with asterisks are used in figure 1, while all 50 points are used in figure 2.)

i		x_i	y_i	z_i	i		x_i	y_i	z_i
1	*	11.16	1.24	22.15	26		3.22	16.78	39.93
2	*	24.20	16.23	2.83	27	*	0.00	0.00	58.20
3		12.85	3.06	22.11	28	*	9.66	20.00	4.73
4	*	19.85	10.72	7.97	29		2.56	3.02	50.55
5	*	10.35	4.11	22.33	30	*	5.22	14.66	40.36
6		24.67	2.40	10.25	31	*	11.77	10.47	13.62
7	*	19.72	1.39	16.83	32		17.25	19.57	6.43
8		15.91	7.74	15.30	33	*	15.10	17.19	12.57
9	*	0.00	20.00	34.60	34	*	25.00	3.87	8.74
10	*	20.87	20.00	5.74	35		12.13	10.79	13.71
11		6.71	6.26	30.97	36	*	25.00	0.00	12.00
12		3.45	12.78	41.24	37		22.33	6.21	10.25
13	*	19.99	4.62	14.72	38		11.52	8.53	15.74
14		14.26	17.87	10.74	39	*	14.59	8.71	14.81
15	*	10.28	15.16	21.59	40	*	15.20	0.00	21.60
16	*	4.51	20.00	15.61	41		7.54	10.69	19.31
17		17.43	3.46	18.60	42	*	5.23	10.72	26.50
18		22.80	12.39	5.47	43		17.32	13.78	12.11
19	*	0.00	4.48	61.77	44	*	2.14	15.03	53.10
20		7.58	1.98	29.87	45	*	0.51	8.37	49.43
21	*	16.70	19.65	6.31	46		22.69	19.63	3.25
22	*	6.08	4.58	35.74	47	*	25.00	20.00	0.60
23		1.99	5.60	51.81	48		5.47	17.13	28.63
24	*	25.00	11.87	4.40	49	*	21.67	14.36	5.52
25	*	14.90	3.12	21.70	50	*	3.31	0.13	44.08

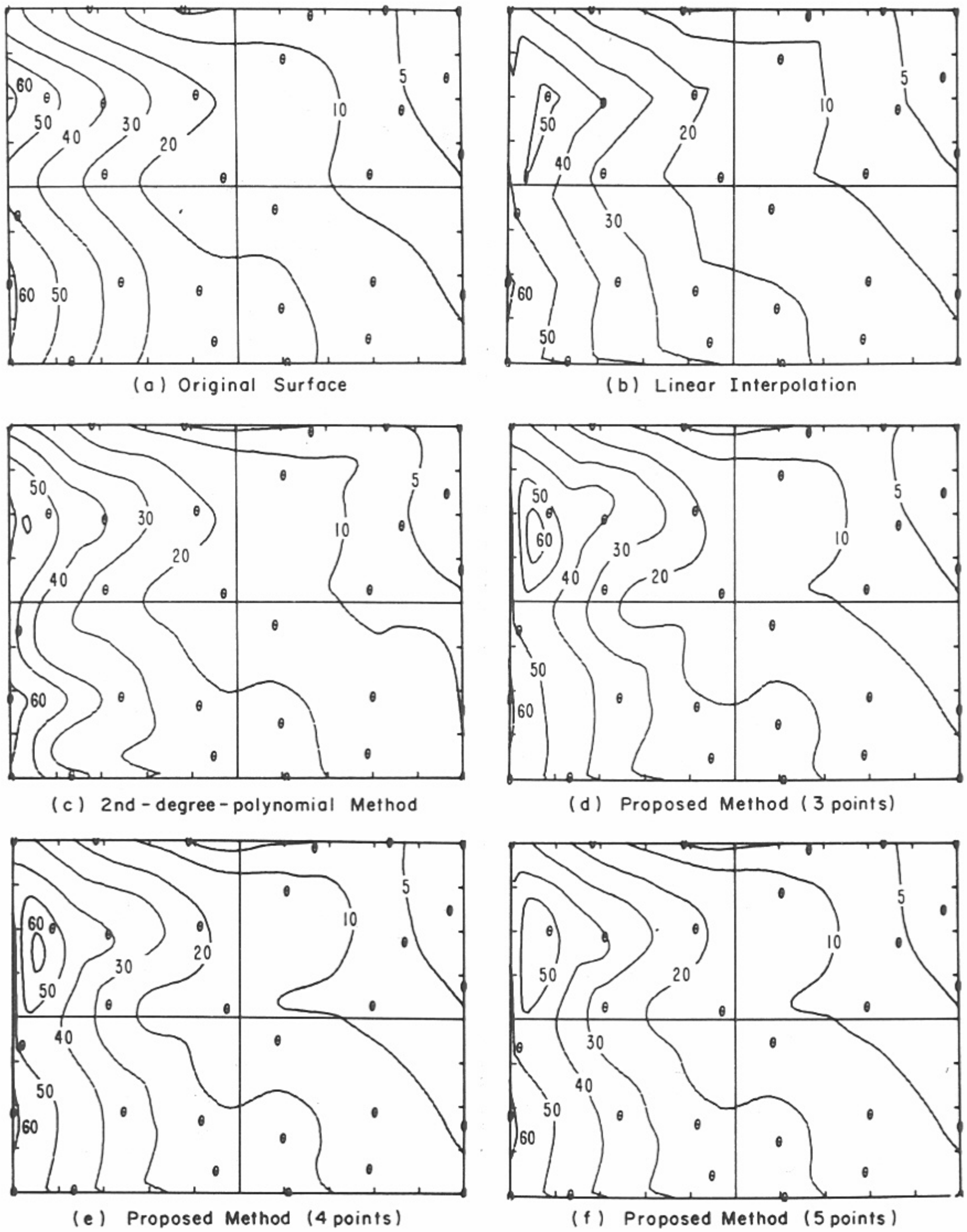

Figure 1. Contour maps for the surfaces fitted to 30 data points given with asterisks in table 1.

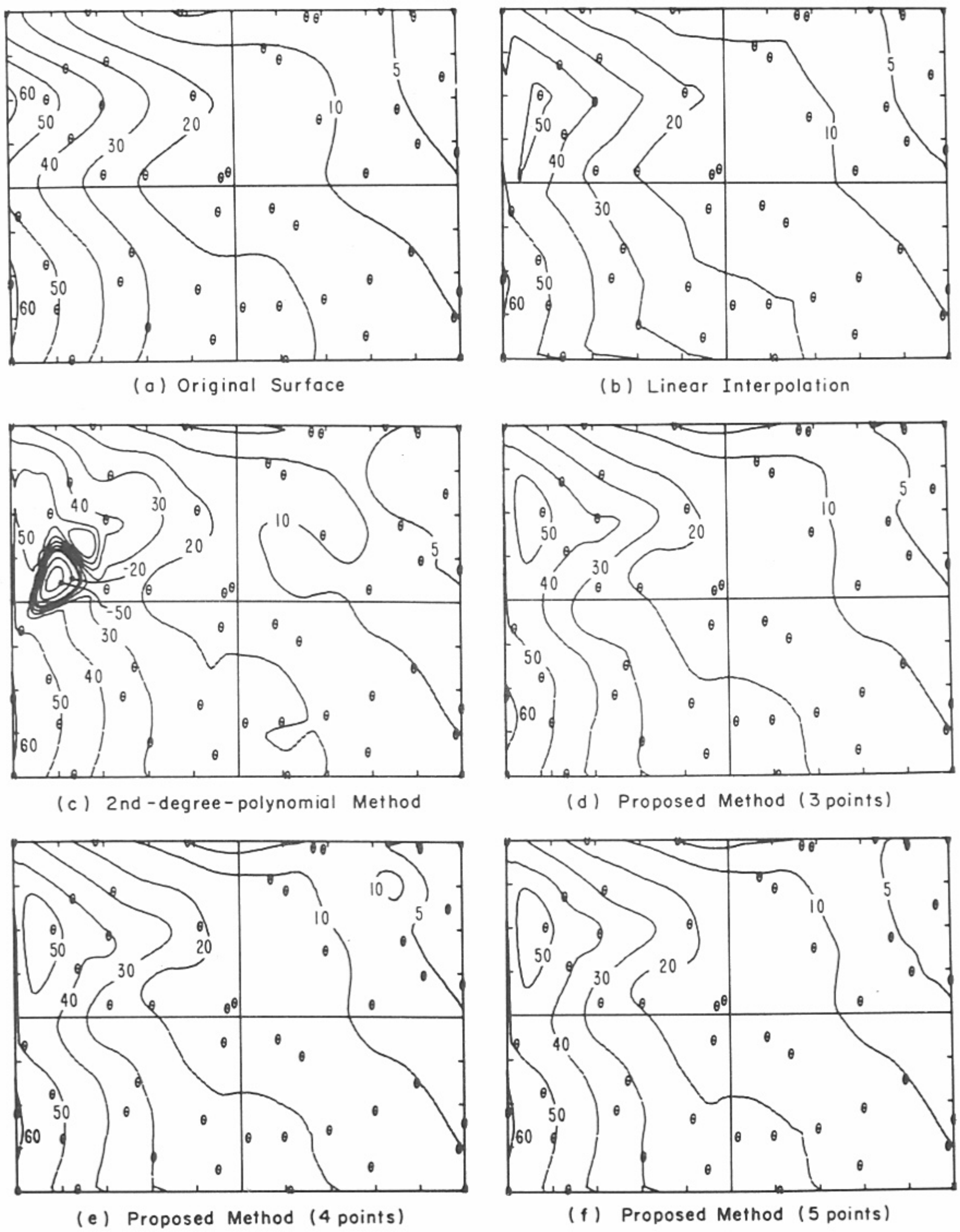

Figure 2. Contour maps for the surfaces fitted to 50 data points given in table 1.

exhibited in the resulting surfaces due to the difference in the number of data points used for the estimation of partial derivatives in the proposed method. Figures 1(c) and 2(c) demonstrate a peculiar idiosyncracy of the method based on second-degree polynomials; more data points yield a much worse result in this example.

Decision as to whether or not the proposed method is applicable to a particular problem rests on each prospective user of the method. The examples given here are expected to aid one in making such a decision. Comparison of (d), (e), or (f) fitted by the proposed method with (a) the original surface or (b) the piecewise-plane surface in each figure should be helpful for such a decision. Also, comparison of figures 1 and 2 gives one some idea on the dependence of the resulting surfaces upon the total number of data points and the complexity of original surfaces.

4. CONCLUDING REMARKS

We have described a method of bivariate interpolation and smooth surface fitting that is applicable when z values are given at points irregularly distributed in an x-y plane. For proper application of the method, the following remarks seem pertinent:

(i) The method does not smooth the data. In other words, the resulting surface passes through all the given points if the method is applied to smooth surface fitting. Therefore, the method is applicable only when the precise z values are given or when the errors are negligible.

(ii) As is true for any method of interpolation, the accuracy of interpolation cannot be guaranteed, unless the method in question has been checked in advance against precise values or a functional form.

(iii) The result of the method is invariant under a rotation of the x-y coordinate system.

(iv) The method is linear. In other words, if $z(x_i, y_i) = a\,z'(x_i, y_i) + b\,z''(x_i, y_i)$ for all i, the interpolated values satisfy $z(x, y) = a\,z'(x, y) + b\,z''(x, y)$, where a and b are arbitrary real constants.

(v) The method gives exact results when $z(x, y)$ represents a plane; i.e., $z(x, y) = a_{00} + a_{10}x + a_{01}y$, where a_{00}, a_{10}, and a_{01} are arbitrary real constants.

(vi) The method requires only straightforward procedures. No problem concerning computational stability or convergence exists in the application of the method.

A computer subprogram package that implements the proposed method is described in Appendix B.

5. ACKNOWLEDGMENTS

The author is grateful to James C. Ferguson of Teledyne-Ryan Aeronautical, San Diego, California, for his information concerning the past works in the subject field. He is also grateful to Rayner K. Rosich and Vincent D. Wayland of the Office of Telecommunications, Boulder, Colorado, for their helpful discussions and critical reviews of this report.