This limiting form of (2.36) is the expression which we shall exploit in the remainder of the study.

The quantity A_∞ appearing in (2.38) is

$$A_\infty(=\gamma_\infty): \ \underline{\text{impulsive index}} \text{ (of the present analysis)*} \qquad (2.39)$$

As we have already noted in our earlier studies [Middleton, 1972b,1973,1974], the Impulsive Index is a measure of the temporal "overlap" or "density", at any instant, of the superposed interference waveforms at the receiver's IF output. It is one of the key parameters of the interference model, in that it critically influences the character of the p.d.'s and P.D.'s of the interference, as observed at the output of the initial (linear) stages of a typical narrow-band receiver. With small values of A_∞ the statistics of the resultant output waveform are dominated by the overlapping of comparatively few, deterministic waveforms, of different levels and shapes, so that the interference has an "impulsive", somewhat structured appearance. For increasingly large values of A_∞ the resultant approaches a normal, or gaussian process, as one would expect from the Central Limit Theorem [Middleton, 1960, Sec. 7.7], as we shall see in more detail later [cf. Sec. 2.4].

2.3 Interference Classes A, B, and C: The Rôle of Input and Receiver Bandwidths:

We are now ready to examine the basic form, (2.38), of $\hat{I}_\infty(r)$ [$=\log \hat{F}_1(ir)$]. The rôle of the duration T_s of a typical emission (as perceived at the output of the ARI (≡ aperture x RF x IF) stages of the narrow-band receiver) is critical in determining the form of $\hat{I}_\infty(r)$.

Let us consider first the important special case when the emission duration T_s is fixed. From Eqs. (2.63a,b), (2.70), (2.72a) of Middleton [1960] we may write for the envelope $\hat{B}_0$, cf. (2.10), (2.30), (2.31a), (2.33),

---

* $A_T(\equiv A_\Lambda A_{\varepsilon,T}=\bar{\nu}_T T$, cf. (2.27c),(2.35)) was designated "impulsive index", A, in the author's earlier treatments [Middleton, 1972b,1973,1974].

$$
\left.
\begin{aligned}
\hat{B}_0(z;\underset{\sim}{\lambda},\underset{\sim}{\theta}') &= \left| \int_{-\infty}^{\infty} E_0(\tau;\underset{\sim}{\lambda},\underset{\sim}{\theta}') e^{-i\Phi_0(\tau;\underset{\sim}{\lambda},\underset{\sim}{\theta}')} h_0(z\bar{T}_s-\tau|\underset{\sim}{\lambda})_{ARI} \right. \\
&\qquad \left. \cdot e^{-i\gamma_0(z\bar{T}_s-\tau)+i\omega_D\tau} \, d\tau \right| \\
&= \left| \int_{-\infty}^{\infty} S_{in}(f';\underset{\sim}{\lambda},\underset{\sim}{\theta}')_0 Y_0(i\omega';\underset{\sim}{\lambda})_{ARI} e^{i\omega' z\bar{T}_s} \, df' \right| \\
&= A_0 e_{0\gamma} |\mathcal{A}_T(\underset{\sim}{\lambda};f_0')| g(\underset{\sim}{\lambda}) u_0(z) \ .
\end{aligned}
\right\} \qquad (2.40)
$$

Here h_0, γ_0 are real, and $h_{ARI}(t) = 2h_0(t)_{ARI} \cos[\omega_0 t-\gamma_0(t)]$ is the weighting function of the composite ARI filter. The (narrow-band) system function Y_0 is obtained from the fourier transform $Y_{0-ARI} = \underset{\sim}{\mathcal{F}}\{h_0 e^{-i\gamma_0}\}$ and ω_D [$=\omega_c-\omega_0$] measures the amount of "detuning" of the input signal (at ω_c, shifted to the IF region) from the (trial) central frequency (f_0) of the ARI stage.

With T_s fixed, we have in general the situation shown in Fig. (2.1)II for the envelope of the narrow-band output of the ARI filter, produced by a typical interference emission of finite duration, T_{in}. The output envelope ($\sim u_0(z)$) produced by a typical input interference envelope [shown as a rectangular pulse in Fig. (2.1)II], always consists of two parts: a part which we shall call Class A with normalized envelope $u_{0A}(z)$, which is produced by the input emission ($\sim E_{0-in}$), which is "on" during the interval $0 \leq z \leq T_{in}$ ($= T_{sA} = \bar{T}_{sA}$); and a part we shall term Class B, with normalized envelope $u_{0B}(z-1)$ ($\neq u_{0A}(z)$), which represents the transient decay of the output of the ARI filter, following the termination of the input emission [$\sim E_0(z)_{in}$]. The sum of Class A and Class B envelopes is called <u>Class C</u>, e.g. $u_{0C}(z) = u_{0A}(z) + u_{0B}(z-1)$, [cf. Fig. (2.1)II, where, of course, $u_{0B}=0$, $z < 1$, $u_{0A} = 0$; $z < 0$, $z > 1$ in our definition. Thus, all receiver outputs are typically Class C, with variable amounts of Class A and Class B, depending on the duration of the typical input interference waveform vis-à-vis the response time of the ARI filter at the front-end of our receiver. Equivalently, the relative extents of the Class A and B components depend generally on the ratio of the bandwidth of the input (Δf_{in}) to the

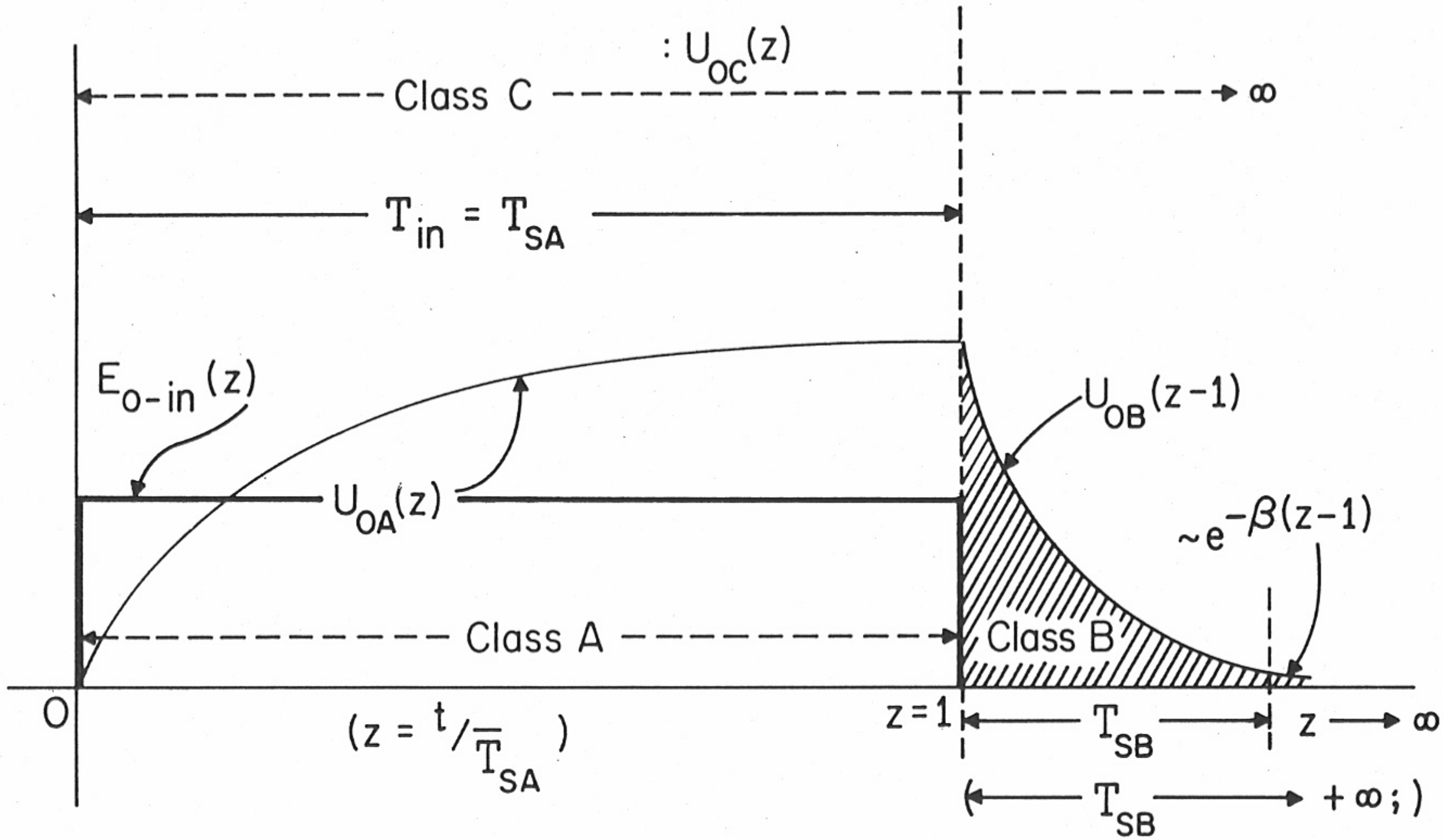

Figure 2.1 (II). Typical output envelope of ARI filter output in (narrowband) receiver, showing Class A and Class B envelope components (positive halves only); Class C = Class A + Class B.

bandwidth Δf_{ARI} of the linear "front-end" of the receiver. With $\Delta f_{in} >> \Delta f_{ARI}$, for example, the usual case of very wide band interference (automobile ignition, fluorescent lights, atmospheric noise), T_{sA} is very small vis-à-vis $\bar{T}_{sB}$: the build-up time ($=T_{in}$) is very brief for Class A, while the duration of Class B depends on the decay time ($\sim \Delta f^{-1}_{ARI}$) of the ARI stage, which is much longer than T_{in}. On the other hand, with narrowband inputs of long duration [$\Delta f_{ARI} >> \Delta f_{in}$], the transient at the termination of the typical input is of negligible effect vis-à-vis the Class A component. For comparable bandwidths ($\Delta f_{ARI} \sim \Delta f_{in}$) both Class A and Class B make comparable contributions, e.g. neither can be ignored vis-à-vis the other, so that we have then generally the Class C waveform in the receiver. [In all cases $\int_0^\infty u_{oA,B}(t)^k dt$, $k > 0$, are finite.]

From (2.38) we see accordingly that $\hat{I}_\infty(r)$ can now be written as the sum of the Class A and Class B components, viz: ($T_{in} = \bar{T}_{sA}$ fixed for the moment):

$$\boxed{\begin{aligned}\hat{I}_\infty(r) &= \nu_\infty\left\{\bar{T}_{sA}\int_0^{z=1}\left\langle J_0(r\hat{B}_{oA})-1\right\rangle_{\underset{\sim}{\theta}',\underset{\sim}{\lambda}} dz + \bar{T}_{sB}\int_0^\infty\left\langle J_0(r\hat{B}_{oB})-1\right\rangle_{\underset{\sim}{\theta}',\underset{\sim}{\lambda}} dz\right\} \\ &= \hat{I}_\infty(r)_A + \hat{I}_\infty(r)_B \equiv \hat{I}_\infty(r)_C\ ,\end{aligned}} \quad \begin{matrix}(2.41)\\(2.41a)\end{matrix}$$

on changing variable $z' \to z$ in $u_{oB}(z'-1) \to u_{oB}(z)$, e.g.

$$\hat{B}_{oA,B} = A_o e_{o\gamma}^{(A,B)}|\mathcal{a}_{RT}(\underset{\sim}{\lambda},f_o')|g(\underset{\sim}{\lambda})(u_{oA}(z),u_{oB}(z)), \tag{2.41b}$$

from (2.40). In terms of the characteristic function (2.19) we see at once that

$$\hat{F}_1(ir) = \hat{F}_1(ir)_A \cdot \hat{F}_1(ir)_B \equiv \hat{F}_1(ir)_C \tag{2.42}$$

with the important result that Class C interference consists of the independent sum of Class A and Class B components, as defined above. Note, also, that the limiting voltages $e_{0\gamma}^{(A)} \neq e_{0\gamma}^{(B)}$, generally, as the receiver responds to "narrow-band" inputs (A) differently from "broad-band" (B).

Specifically, when we can ignore the Class B component [$\bar{T}_{sB}(<\infty) \ll \bar{T}_{sA}$, e.g. sufficiently narrow-band input vis-à-vis the receiver*], we have here, from (2.42), (2.41), in (2.19),

$$\boxed{\begin{aligned}\hat{F}_1(ir)_C \to \hat{F}_1(ir)_A &= \exp\left\{A_{\infty,A}\int_0^1\left\langle J_0(r\hat{B}_{0A})-1\right\rangle_{\underset{\sim}{\theta}',\underset{\sim}{\lambda}} dz\right\} \\ &= e^{-A_{\infty,A}}\cdot\exp\left\{A_{\infty,A}\left\langle J_0(r\hat{B}_{0A})\right\rangle_{z,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\right\},\end{aligned}} \qquad \begin{aligned}&(2.43a)\\&(2.43b)\end{aligned}$$

$$(\bar{\nu}_\infty\bar{T}_{sA}=)\ A_{\infty,A} \gg A_{\infty,B}(=\bar{\nu}_\infty\bar{T}_{sB}), \qquad (2.43b)$$

where the averages $\langle\ \rangle_{z,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}$ are explicitly

$$\left\langle\ \right\rangle_{z,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} = \int_0^1 dz \int_{\Lambda,\underset{\sim}{\theta}'} \frac{w_1(\underset{\sim}{\theta}')\rho(\underset{\sim}{\lambda})}{A_\Lambda}[\ ]\, d\underset{\sim}{\lambda}\, d\underset{\sim}{\theta}' \ . \qquad (2.43c)$$

Similarly, when the Class A component is ignorable [$\bar{T}_{sA} = T_{sin} \ll T_{sB}$ e.g. very broad-band inputs vis-à-vis the receiver's ARI stages*] we get

$$\boxed{\hat{F}_1(ir)_C \to \hat{F}_1(ir)_B = \exp\left\{A_{\infty,B}\int_0^\infty\left\langle J_0(r\hat{B}_{0B})-1\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'} dz\right\},} \qquad \begin{aligned}&(2.44)\\&A_{\infty,B}\gg A_{\infty,A},\end{aligned}$$

with the averages given by (2.43c), (without the average over z). Note that when $r \to \infty$, $\hat{F}_{1-A} \to \exp(-A_{\infty,A})$, while $\hat{F}_{1-B,C} \to 0$. This means, as we shall see in detail in Section 3.1 later, that for Class A interference

---

* The precise conditions for effectively Class A or Class B interference alone are developed in Sec. 7 later.

there will be a non-zero probability of "gaps-in-time", i.e. finite (nonzero) intervals in the receiver's output when there is no waveform present, while for Class B and C interference there is always a nonvanishing waveform level and hence no "gaps-in-time". [Of course, physically there is always some inherent system noise, which makes it strictly impossible to have a true "gaps-in-time" situation.]

We remark, again, that Class A [and consequently Class C] interference models are new. The earlier "classical" analyses [Rice (1945); Middleton (1953); Furutsu and Ishida (1960); Giordano (1970; Giordano and Haber (1972)], for example, all dealt with Class B interference, and for the most part in much less general terms and by different modes of approximation.*

2.3.1 Some Extensions:

Usually, there is an accompanying gaussian background noise, which may arise in a number of ways:

(i). as system noise in the receiver;

(ii). as external interference, which is the resultant of many independent sources, none of which is exceptionally dominant with respect to the others (so that the Central Limit Theorem applies);

(iii). as a mixture of receiver noise and (independent) gaussian external interference. (2.45)

From (2.12b) and the gaussian counterpart of (2.7), viz.

$$\hat{F}_1(i\xi, i\eta)_{X_c, X_s:\text{gauss}} = e^{-(\xi^2+\eta^2)\sigma_G^2/2} \quad , \tag{2.46}$$

we readily find that

$$\hat{F}_1(ir,\phi)_{\text{gauss}} = e^{-\sigma_G^2 r^2/2} = \hat{F}_1(ir)_G; \quad \sigma_G^2 = \sigma_R^2+\sigma_E^2 \quad , \tag{2.47}$$

---

* Technically, Giordano [1970] and Haber [1972] express their results in a Class A format, whenever sample size (T) is finite, cf. remarks in Section (5.3) following.

cf. (2.13), where σ_R^2, σ_E^2 are respectively the receiver and external noise variances.

Applying (2.47) to (2.15), (2.17) shows directly that*

$$W_1(E)_G = E\int_0^\infty rJ_o(rE)e^{-r^2\sigma_G^2/2}dr = \frac{Ee^{-E^2/2\sigma_G^2}}{\sigma_G^2} \quad , \quad E \geq 0 \tag{2.48a}$$

so that

$$D_1(E_o)_G = 1-e^{-E_o^2/2\sigma_G^2} \quad ; \quad P_1(E \geq E_o) = e^{-E_o^2/2\sigma_G^2} \quad , \quad (E_o \geq 0) \quad . \tag{2.48b}$$

As expected, the p.d. and P.D. here are rayleigh.

Our results of Sec. 2.3 above are readily extended to include the more general situation of interference inputs of random duration, e.g. $T_{in} = T_{sA}$ ($\neq \bar{T}_{sA}$) generally. Only the Class A portion of $\hat{I}_\infty(r)$, (2.41), is modified. Letting $z_o \equiv T_{in}/\bar{T}_{sA}$, we have at once, for the desired extension of (2.43),

$$\hat{F}_1(ir)_A = e^{-A_{\infty,A}} \exp\left\{\left\langle\int_0^{z_o} J_o(r\hat{B}_{oA})\,dz\right\rangle_{z_o,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\right\} \quad . \tag{2.49}$$

Combining (2.49) and (2.44) with (2.47) gives us the desired characteristic functions with which we shall be concerned here, and subsequently, in this report:

Class A Interference and Gauss:

$$\hat{F}_1(ir)_{A+G} = e^{-\sigma_G^2r^2/2-A_{\infty,A}} \exp\left\{A_{\infty,A}\left\langle\int_0^{z_o} J_o(r\hat{B}_{oA})dZ\right\rangle_{z_o,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\right\} \tag{2.50}$$

Class B Interference and Gauss:

$$\hat{F}_1(ir)_{B+G} = e^{-\sigma_G^2r^2/2} \exp\left\{A_{\infty,B}\int_0^\infty\left\langle[J_o(r\hat{B}_{oB})-1]\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'}dz\right\} \quad . \tag{2.51}$$

---

* Use Eq. (A.1-49) [Middleton (1960)], for example.

[We shall reserve the analysis for Class C interference and gauss noise, e.g. based here on

$$\hat{F}_1(ir)_{C+G} = \exp\,[-\sigma_G^2 r^2/2 + A_{\infty,A}\left\langle\int_0^{z_0}[J_0(r\hat{B}_{oA})-1]dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}$$

$$+A_{\infty,B}\int_0^{\infty}\left\langle[J_0(r\hat{B}_{oB})-1]\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'}dz] \quad , \tag{2.52}$$

to a subsequent Report.]

2.4 Large Impulsive Indexes:

When the impulsive index, A_∞, is large, we expect asymptotically gaussian statistics for the instantaneous amplitude X [Secs.3,p.26; 5,p. 39, Middleton, 1974], and rayleigh statistics here, cf. (2.48), for the instantaneous envelope E. This latter is easily shown by developing $\hat{I}_\infty(r)$, (2.41) or (2.52), as a power series in r about r = 0, in the usual way.* Thus, the c.f. (2.52) for our general Class C case, with gaussian background noise in addition, becomes

$$\hat{F}_1(ir)_{C+G} = \exp\left\{-\frac{r^2}{2}\sigma_0^2\right\}\cdot\exp\left\{\sum_{n=2}^{\infty}\frac{(-1)^n r^{2n}}{2^{2n}(n!)^2}\left(A_{\infty,A}b_{2n}^{(A)}+A_{\infty,B}b_{2n}^{(B)}\right)\right\}, \tag{2.53}$$

where

$$\sigma_0^2 = (\sigma_R^2+\sigma_E^2)+\bar{\nu}_\infty\bar{T}_{sA}\left\langle\int_0^{z_0}\hat{B}_{oA}^2dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}+\bar{\nu}_\infty\bar{T}_{sB}\int_0^{\infty}\left\langle\hat{B}_{oB}^2\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'}dz, \tag{2.53a}$$

$$b_{2n}^{(A)} = \left\langle\int_0^{z_0}\hat{B}_{oA}^{2n}dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} \tag{2.53b}$$

$$b_{2n}^{(B)} = \int_0^{\infty}\left\langle\hat{B}_{oB}^{2n}\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'}dz \quad . \tag{2.53c}$$

---

* Provided we consider for the moment finite observation intervals $T(<\infty)$, i.e. finite upper limits on the z-integrals in (2.51), (2.52), so that these integrals are uniformly convergent, proper integrals, permitting a series expansion of their integrands. Then, we ultimately have $\hat{F}_1(ir)_{C+G}$: $\lim_{T\to\infty} F_1(ir|T)_{C+G}$, where $(\lim_{T\to\infty})$ is invoked for each term of the resulting expansions. See the comments in B, Sec. (5.2) below.