Section 3.2-A.

An interesting feature of our results here, for Class A interference, is that the c.f., and hence the p.d., and P.D., are not explicitly dependent on the interference source distribution [for the usual Case I, II, and not so common Case III source-receiver conditions, cf. Fig. (2.3)II]. Furthermore, these statistics are insensitive to the propagation law ($\sim \lambda_{max}^{-\gamma}$) involved, which merely affects scale, through the average $\langle \hat{B}_{oA}^2 \rangle = \langle G_o/\lambda^{\gamma} \rangle$. The source distribution does appear, but in averaged form and only in the impulsive index $A_{\infty,A}$, [cf. (2.35), (2.38), (2.39), in conjunction with (2.27)]. Physically, this is understandable, since it is only the average number of emissions per second times the mean duration of these finite emissions (in the receiver, cf. Fig. (2.1)II) to which the receiver can respond. It has no way to distinguish where and in what concentration, or by what propagation law, the sources may be acting [for a given position of the receiver beam $\mathcal{a}_R$, or for any position if $\mathcal{a}_R$ is omnidirectional]. The only thing that counts here in determining the (first-order) statistics of the received input is total input level and process "density", $A_{\infty,A}$. As we shall see in Section (2.7), and in later Sections, this insensitivity to source distribution and propagation law is definitely not characteristic of Class B interference, and, consequently, Class C noise, to the extent that its Class B component is significant.

2.7 The C.F. for Class B Interference:

Here we use (2.67b) for the exponent (2.65). The result is a term like (2.68), plus an additional term for $x_o < x < \infty$, with $\lambda_{max} \geq \lambda \geq 0$, viz:

$$\hat{I}_\infty(r)_B = -A_{\infty,B}\Bigg\{\Bigg\langle\int_0^\infty dz\int_0^{x_o=rG_o/\lambda_{max}^{\gamma}} J_1(x)dx\Bigg\rangle_{\underset{\sim}{\theta}'} + \Bigg\langle\int_0^\infty dz\int_{\Delta(\theta,\phi)} A_{S,V}^{-1}\sigma_{S,V}(\theta,\phi)d(\theta,\phi)\int_{x_o=rG_o/\lambda_{max}^{\gamma}}^\infty J_1(x)dx \cdot\int_{\lambda=(rG_o)^{1/\gamma}/x^{1/\gamma}}^0 \begin{bmatrix} c^2/\lambda^{\mu-1} \\ c^3/\lambda^{\mu-2} \end{bmatrix} d\lambda \Bigg\rangle_{\underset{\sim}{\theta}'}\Bigg\}\;, \tag{2.79}$$

where we have used (2.66) and reversed the order of integration, according to the régime of (iii), Sec. 2.5, and Fig.2.4II) above. Note, in particular, the order of the limits on the λ-integration, which correspond to the variation in x, from $x=x_0$ to $x\to\infty$, cf. Fig.(2.4)II again. [The average over z_0 is unity, as none of the arguments contain z_0; also, $\underset{\sim}{\theta}' = A_0$, $e_{0\gamma}$, etc., as before.] The integrals I_ϕ, $I_{\theta,\phi}$ over ϕ (or θ,ϕ), with $A^{-1}_{S,V}$, become explicitly from (2.62a,b)*

$$\left.\begin{aligned} I_\phi &= \Delta_S/A_S = \left(\frac{2-\mu}{c^2}\right) \lambda_{max}^{\mu-2} , \quad 0 \leq \mu < 2 \\ I_{\theta,\phi} &= \Delta_V/A_V = \left(\frac{3-\mu}{c^3}\right) \lambda_{max}^{\mu-3} , \quad 0 \leq \mu < 3 \end{aligned}\right\} , \tag{2.80}$$

where specifically here

$$A_S = \Delta_S \frac{c^2}{2-\mu} \lambda_{max}^{2-\mu} \quad (0 < \mu < 2); \quad A_V = \Delta_V \frac{c^3}{3-\mu} \lambda_{max}^{3-\mu} \quad (0 < \mu < 3), \tag{2.80a}$$

so that Δ_S, Δ_V are respectively the integrals $\int \sigma_S d\phi$, $\int \sigma_V d\theta d\phi$ in (2.79).

With the above we readily find for (2.79) the (<u>exact</u>) relation

$$\boxed{\hat{I}_\infty(r)_B = -A_{\infty,B}\left\{\left\langle\int_0^\infty dz\int_0^{x_0} J_1(x)dx\right\rangle_{\underset{\sim}{\theta}'} - x_0^\alpha\left\langle\int_0^\infty dz\int_{x_0}^\infty J_1(x)dx/x^\alpha\right\rangle_{\underset{\sim}{\theta}'}\right\}} , \tag{2.81}$$

with

$$x_0 = rG_0/\lambda_{max}^\gamma = r\hat{B}_{0,B} \tag{2.81a}$$

as before, and the new parameter

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* For this Report we shall limit the allowed values of μ as shown in (2.80); extension to other values (μ>2,3) will be considered in a subsequent study, as is the analysis for $\alpha \geq 2$.

$$\alpha \equiv \left.\frac{2-\mu}{\gamma}\right|_{\text{surface}} \quad \text{or} \quad \left.\frac{3-\mu}{\gamma}\right|_{\text{vol.}} \quad . \tag{2.82}$$

This parameter α we call the spatial density-propagation parameter, since it depends on the interacting spatial effects of source density and source propagation law.

The lower limit on α is established by the present condition, i.e., upon the upper limit on μ (=2,or 3), $\gamma > 0$. Analytically, for the integral over x in (2.81) to be convergent, we require that $\alpha > -1/2$. There is, however, no (finite) upper limit on α, so that we can write

$$\boxed{-1/2 < \alpha < \infty} \quad . \tag{2.82a}$$

For the purposes of the present Report we shall, however, further restrict α to the range $(0<\alpha<2)$, which covers many of the practical cases encountered in applications, at least down to quite small values of the exceedance probabilities $P_1(\mathcal{E} < \mathcal{E}_0)$. In a later Report we shall develop the analysis in detail for $(\alpha \geq 2)$.

The first integral in (2.81) is readily evaluated by expanding the Bessel function, followed by termwise integration. We get

$$I_1 = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\left\langle x_0^{2\ell+2}\right\rangle_{z,\underset{\sim}{\theta}'}}{\ell!(\ell+1)!2^{2\ell+1}(2\ell+2)} \quad . \tag{2.83}$$

For the second integration, over x in the second integral of (2.81), we use a Barnes integral representative of $J_1(x)$ [cf. Middleton, 1960, Eq. (13.10)]:

$$J_1(x) = \int_{-\infty i-c}^{\infty i-c} \frac{\Gamma(-s)}{\Gamma(s+2)} \left(\frac{x}{2}\right)^{2s+1} \frac{ds}{2\pi i} \quad , \quad c > 0 \; , \tag{2.84}$$

so that

$$I_\alpha \equiv \left\langle x_0^\alpha \int_{x_0}^{\infty} \frac{J_1(x)}{x^\alpha}\, dx \right\rangle_{z,\underset{\sim}{\theta}'} = \left\langle x_0^\alpha \int_{-\infty i-c}^{\infty i-c} \frac{\Gamma(-s)ds/2\pi i}{\Gamma(s+2)2^{2s+1}} \int_{x_0}^{\infty} x^{2s+1-\alpha} dx \right\rangle_{z,\underset{\sim}{\theta}'} . \tag{2.85a}$$

This becomes, on choosing $c = -5/4$ (as a result of the condition $\alpha > -1/2$) and carrying out the x-integration

$$\left. \frac{x^{2s+2-\alpha}}{2s+2-\alpha} \right|_{x_0}^{\infty} = \frac{-x_0^{2s+2-\alpha}}{2s+2-\alpha} \; ; \; \begin{cases} \mathrm{Re}(2s+2-\alpha) < 0, \text{ or} \\ s < \frac{\alpha-2}{2} . \end{cases} \tag{2.85b}$$

The resulting integral I_α is now explicitly

$$I_\alpha = \left\langle x_0^\alpha \int_{-\infty i-c}^{\infty i-c} \frac{-\Gamma(-s)x_0^{2s+2-\alpha} ds/2\pi i}{\Gamma(s+2)2^{2s+1}(2s+2-\alpha)} \right\rangle_{z,\underset{\sim}{\theta}'} , \tag{2.85c}$$

which has a simple pole at $s = (\alpha-2)/2$ $(>-5/4)$, and at $s=0,1,2,3,...$ $(0 < \alpha < 2)$. The residue at $s = (\alpha-2)/2$ is $-\Gamma(1-\alpha/2)/2^{\alpha-1}\Gamma(1+\alpha/2)$, while those of Γ-function are $(-1)^\ell/\ell!$ at $s=\ell$ $(=0,1,2,..)$. The result is

$$I_\alpha = \frac{-\Gamma(1-\alpha/2)\langle x_0^\alpha \rangle}{2^{\alpha-1}\Gamma(1+\alpha/2)} - \sum_{\ell=0}^{\infty} \frac{(-1)\,\langle x_0^{2\ell+2} \rangle}{\ell!(\ell+1)\; 2^{2\ell+1}(2\ell+2-\alpha)} . \tag{2.85d}$$

The exponent $\hat{I}_\infty$ of the c.f. for Class B interference thus becomes, on combining I_1, (2.83), and I_α, (2.85d), in (2.81):

$$\hat{I}_{\infty,B}(r) = -A_{\infty,B} \left\{ \frac{\Gamma(1-\alpha/2)\langle x_0^\alpha \rangle}{2^{\alpha-1}\Gamma(1+\alpha/2)} + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \langle x_0^{2\ell+2} \rangle}{\ell!(\ell+1)!2^{2\ell+1}} \left[\frac{4\ell+4-\alpha}{(2\ell+2)(2\ell+2-\alpha)} \right] \right\}, \quad (0 < \alpha < 2) , \tag{2.86}$$

with $\langle\ \rangle = \langle\ \rangle_{z,\underset{\sim}{\theta}'} \equiv \langle \int_0^\infty (\ )dz \rangle_{\theta'}$, etc.

With the additive, accompanying gaussian background, cf. Sec. (2.3.1), we have at last the desired c.f.:

$$\boxed{\hat{F}_1(ir)_{B+G} = \exp\left\{-b_{1\alpha}A_{\infty,B}r^{\alpha} - (\sigma_G^2+b_{2\alpha}A_{\infty,B})r^2/2 - \sum_{\ell=1}^{\infty}(-1)^{\ell}b_{(2\ell+2)\alpha}A_{\infty,B}r^{2\ell+2}\right\}, (0 < \alpha < 2),} \tag{2.87}$$

which, like (2.81), is also exact, so far. Here we have explicitly

$$\left\{\begin{array}{ll} b_{1\alpha} \equiv \dfrac{\Gamma(1-\alpha/2)}{2^{\alpha-1}\Gamma(1+\alpha/2)}\left\langle G_{o,B}^{\alpha}\right\rangle\lambda_{max}^{-\alpha\gamma} = \dfrac{\Gamma(1-\alpha/2)}{2^{\alpha/2-1}\Gamma(1+\alpha/2)}\left\langle\left(\dfrac{\hat{B}_{o,B}}{\sqrt{2}}\right)^{\alpha}\right\rangle(>0) & (2.87a) \\ b_{2\alpha} \equiv \left(\dfrac{4-\alpha}{2-\alpha}\right)\left\langle G_{o,B}^{2}\right\rangle\lambda_{max}^{-2} = \left(\dfrac{4-\alpha}{2-\alpha}\right)\dfrac{\left\langle\hat{B}_{o,B}^{2}\right\rangle}{2} \quad (>0), & (2.87b) \\ b_{(2\ell+2)\alpha} \equiv \dfrac{(4\ell+4-\alpha)}{\ell!(\ell+1)!(2\ell+2-\alpha)(2\ell+2)}\dfrac{\left\langle\hat{B}_{o,B}^{2\ell+2}\right\rangle}{2^{2\ell+1}} \quad (>0);\ (\hat{B}_{o,B} = G_{o,B}/\lambda_{max}^{\gamma}), & (2.87c) \end{array}\right.$$

since $-0 < \alpha < 2$, and from (2.65) we write

$$\left\langle G_{o,B}^{\alpha}\right\rangle = \left\langle e_{o\gamma}^{(B)\alpha}\right\rangle\left\langle A_o^{\alpha}\right\rangle\left\langle|\mathcal{a}_{RT}|^{\alpha}\right\rangle_{\mathcal{a}}\left\langle g_{S,V}^{\alpha}\right\rangle(4\pi c)^{-\alpha\gamma}\int_0^{\infty}u_o(z)_B^{\alpha}dz \quad (>0), \tag{2.87d}$$

and formally $\left\langle G_{o,B}^{2\ell+2}\right\rangle$ is given by (2.87d) on replacing α by $2\ell+2$, etc.

A. The Approximating C.F.'s for $(0 < \alpha < 2)$:

Unlike Class A interference [Sec. 2.6], where the c.f. is solely a function of r^2 [cf. (2.76)-(2.78)], and where a single "steepest-descent" approximation [cf.(2.72)-(2.76)] provides a good fit for both large, small, and intermediate values of r^2 (and hence for E_o,E), Class B noise requires a pair of approximating c.f.'s, one of which will at least insure suitably bounded behaviour of the exceedance probability P_1 $(E > E_o)$ as $E_o \to \infty$, including the existence of all finite moments of the envelope $\langle E^{\beta}\rangle (0<\beta<\infty)$, and the other of which will provide a satisfactory account of P_1 for small and intermediate values of E ($>E_0$). It is the presence of the term $O(r^{\alpha})$ in the (exponent of the) c.f. (2.87) for Class B interference, in addition to the typical development in powers of r^2 (analogous to that for the Class A noise), which forces this double approximation for our canonical c.f.'s, and P.D.'s, $P_1(E>E_0)$, pdf's, $w_1(E)$, here.

At this point we define the gaussian variance

$$\Delta\sigma_G^2 \equiv \sigma_G^2 + b_{2\alpha}A_{\infty,B} = b_{2\alpha}A_{\infty,B}(1+\sigma_G^2/b_{2\alpha}A_{\infty,B}) \tag{2.88a}$$

$$\equiv \Omega_{2B}^{(G)}(1+\Gamma_B^{(G)}), \tag{2.88b}$$

with

$$\Omega_{2B}^{(G)} \equiv b_{2\alpha}A_{\infty,B} \quad ; \Gamma_B^{(G)} \equiv \sigma_G^2/\Omega_{2B}^{(G)} = \frac{\text{indep. gauss intensity}}{\text{"impulsive" gauss Intensity}} \tag{2.88c}$$

$$= \left(\frac{4-\alpha}{2-\alpha}\right)\Omega_{2B}, \text{ cf. Eq. (3.2a)ff.}$$

where $\Omega_{2B}^{(G)}$ is the "impulsive" contribution to the gaussian component arising from the Class B noise alone, and where $\sigma_G^2(=\sigma_E^2+\sigma_R^2)$ are the (independent) inherently gaussian contributions from potential external (gaussian) sources and from the receiver noise (essentially all arising in the initial linear input stages), cf.(2.47). (Note that $\Omega_{2B}^{(G)},\Gamma_B^{(G)}$ are also functions of α here.)

For the c.f. which is appropriate to the intermediate range of envelope values, including the very small ($E,E_0\to 0$), the controlling term in the exponent of the (exact) c.f. (2.87) is <u>the smallest power of r with negative coefficient</u>, e.g., $-b_{1\alpha}A_{\infty,B}r^{\alpha}$ here, so that this approximate form remains a proper c.f., e.g. $\lim_{r\to 0}\hat{F}_1 = 1$, $\lim_{r\to\infty}\hat{F}_1\to 0$. The form of the associated pdf. and P.D. for small and moderate values of E, E_0 is governed principally by the behaviour of the c.f. as r becomes large. Thus, as a first approximation which ignores any gaussian contributions, we have from (2.87)

$$\boxed{\hat{F}_1(ir)_B' \doteq e^{-b_{1\alpha}A_{\infty,B}r^{\alpha}} \quad , \qquad 0 < \alpha < 2.} \tag{2.89}$$

However, practically there is always at least an observeable gaussian system noise component, and as noted above, cf. (2.87), (2.88a), an additional gaussian term ($\sim b_{2\alpha} A_{\infty,B}$) contributed by the "impulsive" Class B noise, so that the more realistic intermediate c.f. here is now

$$\boxed{\hat{F}_1(Ir)_{(B+G)-I} \doteq e^{-b_{1\alpha} A_{\infty,B} r^{\alpha} - \Delta\sigma_G^2 r^2/2}} \quad , \; (0 < \alpha < 2), \tag{2.90}$$

where the subscript (-I) indicates the c.f. for the range ($0 \leq E, E_0 \leq E_B$) of envelope values. (The precise definition of E_B will be given presently, cf. Sec. 3.2).

For values of $E, E_0 > E_B$ we require a c.f. approximating the exact relation (2.87) where the largest (r-dependent) contribution to the exponent about r = 0 and in the finite (nonzero) neighborhood of r=0 is the controlling term. For this we seek again a "steepest descent" form for the exponent of (2.87), exclusive of the term in r^{α}, which as we shall see below is always here smaller than the former (for $0 \leq r \leq \varepsilon$) and thus does not control the character of the c.f. at small r (and hence for large E, E_0). Accordingly, as in the Class A cases, cf. Section 2.6 above, we wish to represent the class B terms (exclusive of r^{α}) in (2.87) by a series of the form

$$A_{\infty,B} - b_{2\alpha} A_{\infty,B} \frac{r^2}{2} + \sum_{\ell=1}^{\infty} (-1)^{\ell+1} A_{\infty,B} b_{(2\ell+2)\alpha} r^{2\ell+2} = Ae^{-ar^2} \left[1 + \sum_{k=1}^{\infty} B_k r^{2k}\right], \tag{2.91}$$

where the "steepest-descent" nature of the approximation is exhibited not only by the exponential factor but by requiring the vanishing of the B_1-term in the right hand series, where the nearest "correction" term (k=2) is $O(r^4)$ and quite ignorable vis-à-vis unity. This condition and a term by term comparison of (2.91) determine all the parameters A, a, B_k ($k \geq 2$), which are readily found to be

$$\left.\begin{aligned} &A = A_{\infty,B} \; ; \; a = b_{2\alpha}/2 \quad ; \quad (B_1 = 0) \\ &\qquad\qquad B_2 = b_{4\alpha} - b_{2\alpha}^2/8 \\ &\qquad\qquad B_3 = b_{6\alpha} + \frac{b_{4\alpha}b_{2\alpha}}{2} + \frac{b_{2\alpha}^3}{48}, \text{ etc.} \end{aligned}\right\} . \tag{2.92}$$

Clearly, $A_{\infty,B}e^{-ar^2}$ dominates $-A_{\infty,B}b_{1\alpha}r^{\alpha}$, $A_{\infty,B}e^{-ar^2}r^4$, etc., at and in the neighborhood of $r = 0$, and this is the determining element for this approximation to the exact c.f. (2.87).

Accordingly, we have finally for the c.f. appropriate at least to the large values of E, E_0, i.e. for the "rare events",

$$\boxed{\hat{F}_1(ir)_{(B+G)-II} \doteq \left\{e^{-A_{\infty,B}}\exp\left[A_{\infty,B}e^{-b_{2\alpha}r^2/2} - \sigma_G^2 r^2/2\right]\right\}[1+0(r^{\alpha},r^4)], \quad (0 < \alpha < 2).} \tag{2.93}$$

Comparison with (2.77) shows at once that this approximate c.f. for Class B interference has the same (approximate) form as that for Class A noise, and thus will yield the same type of pdf's and P.D.'s, etc., cf. Sec. (3.1). As we shall see later, in Section 4, this has the important consequence of insuring that all (finite) moments of the envelope, $\langle E^{\beta}\rangle$, exist, as required by the physics of the situation in all cases.

We note in passing that a more elaborate approximation to $\hat{F}_1$ here may be obtained by a combination of (2.89) and (2.93), viz:

$$\hat{F}_1(ir)_{(B+G)-III} = e^{-b_{1\alpha}A_{\infty,B}r^{\alpha} - A_{\infty,B} - \sigma_G^2 r^2/2} \exp\left(A_{\infty,B}e^{-b_{2\alpha}r^2/2}\right), \tag{2.94}$$

which may be used for intermediate ranges of r for improved fits to the corresponding intermediate ranges for the envelope. However, since the resulting pdf's, and P.D.'s, are considerably more complex and since the simpler forms of c.f. above, e.g. (2.90), (2.93), appear ultimately to provide excellent agreement with observation, we shall not pursue the consequences of using (2.94) further in the present Report.

We remark that both approximating c.f.'s (2.90), (2.93) for the true Class B c.f. are such as to give pdf's which are not properly normalized; each pdf, $w_1(\mathcal{E})_{B-I}$, $w(\mathcal{E})_{B-II}$, (4.3),(4.4) does not yield $\langle \mathcal{E}^2 \rangle_B = 1$. The former gives an infinite value, while the latter, although Class A-type, cf. (2.78), yields $\langle \mathcal{E}^2 \rangle_{B-II} = 4G_B^2$ ($\neq 1$), where G_B is given by (3.12b). Thus, $w_1(\mathcal{E})_{B-II,norm} = (4G_B^2)^{-1} w_1(\mathcal{E})_{B-II}$, while the normalization of $w_1(\mathcal{E})_{B-I}$ requires, instead, a change of scale for the argument $\mathcal{E}$ (and $\therefore$ $\mathcal{E}_0$ in the associated PD). How this is done is described in Section 3.2-A.

Finally, it is important to observe that unlike the Class A interference discussed in Section 2.6 above, the (first-order) statistics of Class B noise are obviously sensitive to the combined effects of source distribution (μ) and propagation law (γ), through the density-propagation parameter α, cf. (2.82). Physically, this may generally be explained by the fact that now the receiver itself largely determines the waveform of its response to the (relatively) short input excitations, unlike Class A noise, where apart from amplification (for fixed aperture bearing) the receiver negligibly influences the structure of the received wave trains. The composite sum of the "tails" of the transients in the receiver, generated by the Class B input, depend on the (relative) times of arrival of individual wave trains ($\sim$source distribution) and on the level of the various wavetrains ($\sim$source distribution and propagation law). The (relatively) longer time-pedestal provided by the transient decay of individual impulses provides a wider range of possible total amplitudes of overlapping transients and hence a more gradual transition to given thresholds (E_0) of the exceedance probabilities $P_1(E > E_0)$, than that occuring with Class A interference, as can be seen subsequently in Figs.(3.5, 3.6)II vs. Figs.(3.1)II,(3.2)II. [These effects accordingly influence the instantaneous waveform in the receiver's ARI stage, and hence the statistics of that waveform.] In any case, the sensitivity to α is thus a receiver bandwidth phenomenon, which is illustrated by the experimental and theoretical results shown in Part I of this Report.