APPENDIX. AN APPROXIMATION TO THE NOISE PROBABILITY DENSITY FUNCTION

As we have seen, we are confronted with the noise probability density function,

$$p_Z(z) = \sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-\frac{z^2}{2\sigma_m^2}} , \qquad \text{(A1)}$$

where

$$\sigma_m^2 = \frac{m/A + \Gamma'}{1 + \Gamma'} .$$

The optimum detection algorithms and the numerical computations required the use of this $p_Z(z)$. It is of use to develope a simpler expression which is still a good approximation to the above and which can be used then to develop good suboptimum receivers or which can be used perhaps to simplify future numerical calculations.

We first note that each term of the sum in (A1) is a Fourier transform, so that, with a change of integration and summation, we get

$$p_Z(z) = \frac{e^{-A}}{2\pi} \int_{-\infty}^{\infty} e^{j\xi z} \left(\sum_{m=0}^{\infty} \frac{A^m}{m!} e^{-\frac{\xi^2\sigma_m^2}{2}} \right) d\xi . \qquad \text{(A2)}$$

Therefore, we have

$$p_Z(z) = \frac{e^{-A}}{2\pi} \int_{-\infty}^{\infty} e^{j\xi z} e^{-\frac{\xi^2}{2}\left(\frac{\Gamma'}{1+\Gamma'}\right)} \sum_{m=0}^{\infty} \frac{\{A \exp[-\frac{\xi^2}{2} \frac{1}{A(1+\Gamma')}]\}^m}{m!} d\xi . \qquad \text{(A3)}$$

or

$$p_Z(z) = \frac{e^{-A}}{2\pi} \int_{-\infty}^{\infty} e^{j\xi z} e^{-\frac{\xi^2}{2}\left(\frac{\Gamma'}{1+\Gamma'}\right)} \exp\left[Ae^{-\frac{\xi^2}{2A(1+\Gamma')}} \right] d\xi. \quad \text{(A4)}$$

In (A4), the exp[·] part of the integrand is well behaved for the small values of A and Γ' of interest to us and easily approximated. We use

$$\exp\left[Ae^{-\frac{\xi^2}{2A(1+\Gamma')}} \right] \approx 1 + e^{-\frac{\xi^2}{2A(1+\Gamma')}} (e^A - 1). \quad \text{(A5)}$$

Note that our approximation is an upper bound, matches at and near the origin, and also for ξ large. The maximum difference occurs at

$$\exp\left[\frac{-\xi^2}{2A(1+\Gamma')} \right] = \frac{1}{A} \ln \frac{e^A - 1}{A} , \quad \text{(A6)}$$

and this maximum difference is exceedingly small for small A and Γ'. For example, for $A = 0.35$ and $\Gamma' = 0.5 \times 10^{-3}$, this maximum difference is 0.018, or less than 1 percent. The bound is even tighter for smaller values of A. Using (A6) in (A4), we obtain

$$p_Z(z) \approx \frac{e^{-A}}{\sqrt{2\pi\beta^2}} e^{-\frac{z^2}{2\beta^2}} + \frac{\alpha e^{-A}}{\sqrt{2\pi\gamma^2}} e^{-\frac{\zeta^2}{2\gamma^2}} , \quad \text{(A7)}$$

where

$$\alpha = e^A - 1 ,$$

$$\beta^2 = \frac{\Gamma'}{1+\Gamma'} ,$$

and

$$\gamma^2 = \frac{A\Gamma'+1}{A(1+\Gamma')}$$

This approximation is a proper pdf; i.e., it integrates to 1, and computations show that it begins to differ slightly from $p_Z(z)$, for small A and Γ', only for values of z greater than 10, i.e., after 10 standard deviations.

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Since communications systems are seldom interfered with by classical white Gaussian noise, Middleton's recently developed physical-statistical model of impulsive interference is applied to real world communications channels.

The main impulsive interference models that have been proposed to date are summarized and Middleton's model is specified in some detail, giving the statistics required for the solution of signal detection problems. Excellent agreement of these statistics with corresponding measured statistics is shown.

Middleton's model for narrow-band impulsive interference (a subset of the overall model) is applied to a class of optimal signal detection problems. Optimum detection algorithms are given for coherent and incoherent binary detection. The three basic digital signaling waveforms are considered; i.e., antipodal, orthogonal

(continued)

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and ON-OFF keying. Performance bounds are obtained for these signalling situations. Since it is known that in order to gain significant improvement over current receivers, the number of independent samples of the received interference waveform must be large, the performance results are given parametrically in the number of samples, or equivalently, the time-bandwidth product. Performance of the current suboptimum receivers is obtained and compared to the optimum performance. It is shown that substantial savings in signal power and/or spectrum space can be achieved.

Since physical realization of the completely optimum detection algorithms cannot, in general, be economically obtained, the corresponding locally optimum or threshold receivers are derived and their performance given. These threshold receiver structures are canonical in nature in that their structure is independent of the form of the interference. They are also adaptive in that they must be able to adjust to the changing interference environment. Locally optimum structures are given here for coherent and incoherent detection with constant signal levels and various kinds of fading. The case in which phase estimation is used (partially coherent reception) is also considered.

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